Early Childhood Research and Education: An Inter-theoretical Focus 7

Aleksander Veraksa
Yulia Solovieva *Editors*

Learning Mathematics by Cultural-Historical Theory Implementation

Understanding Vygotsky's Approach

Springer

Early Childhood Research and Education: An Inter-theoretical Focus

Volume 7

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Chapter 1
Introduction

Yulia Solovieva, Aleksander Veraksa, and Yury Zinchenko

The present book is dedicated to the topic of acquisition of mathematical abilities during the process of ontogenetic development. The theme of acquisition of general cultural experience was in the focus of cultural-historical theory, starting from works of L. S. Vygotsky. According to cultural-historical approach, mathematical knowledge and mathematical abilities are not isolated abilities, but an integrated part of this general cultural-historical experience. L. S. Vygotsky saw the specificity of child development in the child's mastery of cultural forms of psychological activity, which appears twice on the scene of the child's psychological development: firstly, as external activity, and later, as internal, or internalized mental activity. The essential point of this paradigm is that there might be no kind of internalized mental activity if no kind of external activity would be introduced in the child's life experience.

The mathematics course traditionally occupies a central place in the curriculum of schools and universities. Learning mathematics in school has an impact on the daily life of every member of society. Mathematics forms a culture of thinking and personality traits. It is important to understand that ignoring the problems of teaching mathematics in a modern school can cause great damage to the technical and economic development of society. In the logic of the cultural-historical approach, as well as the theory of activity, the issue of studying the difficulties in mastering the conceptual content of educational subjects has a long tradition—this topic was given considerable attention by L. S. Vygotsky, A. N. Leontiev, D. B. Elkonin, P. Ya. Galperin, V. V. Davydov, N. F. Talyzina, and others. Within the framework of these approaches, an idea and understanding of how the process of teaching mathematics should develop

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is formed. Among the important aspects, the need to develop students' personalities through educational means, reliance on scientific, theoretical concepts, meaningfulness, and awareness of learning is highlighted, and the result of learning should be qualitative changes in the content and structure of students' consciousness. Factorial approach, aimed at statistical calculation of different variable influence on child development (personal characteristic, components of environment, etc.), was actively criticized at the dawn of cultural-historical psychology. It is possible to statistically quite accurately calculate the degree of "influence" on a child's academic performance by his weight at birth or the presence of physical warm-up at the beginning of each lesson, but in this case we remain at the level of empirical connections, which not only do not explain anything, but also create false attitudes among teachers. The "trap" of the factorial approach is that we are not immune to identifying any new constructs that could potentially influence academic performance, as illustrated by the emergence of, for example, "math anxiety." This approach can be overcome only if we do not limit ourselves to the empirical level of study, but move to the theoretical level, and instead of analyzing "phenomenon–properties–factors," we examine the essence of what is being studied and reproduce the conditions for the formation of a certain phenomenon in reality. The real problem facing science and teaching practice, which is obscured by the pseudoscientific concepts of "mathematics anxiety," is the problem of the causes of difficulties experienced by children, starting from the primary school period, in particular, when learning mathematics. Of great concern in this regard are studies that suggest a genetic and biological basis for "math anxiety" (Luttenberger, 2018; Júlio-Costa et al., 2019; Carvalho et al., 2022). In the cultural-historical concept, a special research method was proposed for these purposes—a formative experiment (Gal'perin, 2002). So, for example, from the standpoint of P. Ya. Galperin, the problem of underachievement (including in mathematics) is associated either with the formation in students of the content of actions that is inadequate to mathematical concepts, or with insufficient development of the properties of the necessary actions (reasonableness, consciousness, generality, etc.). P. Ya. Galperin in his research showed that it is precisely in low-achieving students that, through the effective formation of the necessary actions, it is possible to remove negative emotions and create a positive attitude toward the learning process (Gal'perin, 2002). By highlighting "mathematical anxiety" as a special "reason" for academic failure, we are actually replacing the search for causes with a search for empirical relationships, which is unproductive for both science and practice. A number of studies also note that "math anxiety" is often a consequence of a person's general anxiety, which is caused by the characteristics of the nervous system, manifestations of social anxiety, or fear of failure (Luttenberger et al., 2018). Also, some researchers identify other types of subject anxiety, in particular "chemical anxiety" (Bowen, 1999), "foreign language learning anxiety" (Matsuda & Gobel, 2004), "scientific anxiety" (Mallow et al., 2010), etc. However, it is important to note that, despite the identification of these types anxiety, the basis for their development is the general level of anxiety, which increases in the most vulnerable situations for a person.

From the position of cultural-historical psychology emotional experience ("perezhivanie") is the basis for the perception of the social situation of development (Fleer et al., 2017). It can be argued that the absence of emotional experience means the absence of development. Since the developmental situation is social, the emotional experience determines the process of interaction of the child with others. In this sense, emotional experience about mathematics can be understood as a result of living in a social developmental situation initiated by an adult in the process of interaction with a child. The specificity of interaction gives rise to the corresponding psychological education, since, according to the basic genetic law of cultural development, "Every function in the cultural development of a child appears on the scene twice, on two levels, first social, then psychological, first between people, as an interpsychic category, then inside the child, as an intrapsychic category" (Vygotsky, 1983, p. 145). Thus, there is every reason to consider ideas about anxiety in relation to specific content as the result of one of the variants of the social development situation, which has developed largely under the influence of adults who attach particularly important importance to the successful development, in this case, of mathematical content. It follows from the law that initially anxiety about mathematics exists on the external plane as a form of interaction between participants in a social situation of development, which subsequently passes into the internal plane, preventing the development of mathematics. It is important to note that both children and adults often experience difficulties in solving not only complex, but also simple mathematical problems, and it is clear that this situation is experienced as a situation of failure. However, in this case, what and how to change in teaching so that the task becomes accessible to children, regardless of whether they have "mathematical anxiety" or even mathematical abilities is the most important. The mass school program is available to all normally developing children without exception, which has been repeatedly shown in the studies of N. F. Talyzina.

L. S. Vygotsky has emphasized one important feature of the process of psychological development as a paradox of child development. This process turns out in a way that what should appear at the end of development is already present in it at the beginning of development in the general human culture and has to be shared between the child and the adult. The acquisition of the ideal form of cultural activity is carried out in the process of interaction between the primary form of the child, as the child's external activity and the ideal form, as a kind of activity, existent within the culture. The bearer of this cultural interaction is the adult. In this case, the source of development is not environment directly, but the optimum way of organization of the child's activity by an adult. In this way, the adult would guarantee an ideal form of environment to the child.

In his works, L. S. Vygotsky stated that if there is no corresponding ideal form in the environment, then development will be limited. As an example of such ideal kind of organization of activity, the process of teaching and learning of mathematics should be studied. Adult mathematical thinking can be understood as an ideal form of cultural activity, which is previous form of child's acquisition of this activity. Adult's activity of organization of the process of teaching and learning is an important component of child development, the mastery of which occurs.

The present book is based on the growing interest of international academic community to Vygotsky's approach in psychology and education, that is, in all sciences directed to the study of human development. This interest of the specialists from different countries indicates the necessity of implementation of new innovative solution of the problems of teaching and development. Traditionally, these problems were studied in the works of J. Piaget and continued by constructivist approach in general. This approach doesn't correspond to the essential proposals of the followers of cultural-historical approach. Constructivist approach claims for development as spontaneous process, while cultural-historical Vygotsky's approach claims for development as organized cultural activity, shared between adult and child.

The present book provides theoretical and methodological considerations, together with the results of the research provided by psychologists and teachers from different countries, all of them interested in the application and development of cultural-historical approach. The research presented in the book is aimed to different aspects of acquisition of mathematical concepts and abilities in preschool period and in the further stages of educational process. The chapters of the book present the usage of different cultural symbolic means in the frame of cultural-historical approach as well as of broad issues of teachers' education in mathematics. Different chapters include consideration, comparison, and critical discussion of present programs for teaching of mathematics at different education levels: preschool age, primary school, middle, and high school. Researchers from different countries, as participants in the book have expressed different positions regarding the necessity of early preparation for introduction of mathematical thinking. Some chapters are devoted to the topic of the work with psychological content of preparation for teaching at school, such as imagination, creativity, voluntary activity, and reflective use of symbolic means. The authors provide comparison of cultural-historical understanding of preparation for school in comparison with constructivist approach.

Special attention is given to concrete examples of work, educational programs used that makes possible to replicate the results in various settings. The conditions, means, and predictors of proper acquisition of mathematical concepts at different ages and different educational levels (preschoolers, primary, and middle secondary education) are discussed. Results of longitudinal studies of the development of mathematical abilities in children of the first three grades of primary school, studying according to different mathematical programs show that self-regulation turns out to be a significant factor for the selection of programs involving either an increased level of complexity or an intensification of traditional content for the formation of mathematical skills.

One line of research highlights the necessity of the cultural-historical approach to the acquisition of concepts based on introduction and development of logical actions. J. Piaget considered logical concepts as naturally emerging operations. The development of mathematical thinking in school is closely connected to "natural" appearance of these operations in children. The authors of the chapters of the book examine the early stages of introduction of symbolic and logical actions, starting from preschool age. The chapters explore the formation of multiplicative concepts and the introduction of special modeling means to the inquiry-based learning in order to promote students' actions to the conceptual level of orientation in the coordination of two magnitudes' changes. The authors also describe the principles of digital support design, which aims to scaffold the formation of multiplicative concepts at the very first stages of math education. The cultural activity view on a psychological study is presented as the necessity and possibility of organized and orientated introduction and formation of intellectual actions, which can replace observation of these operations' spontaneous development.

The majority of the chapters of the book are written from the point of view of cultural-historical approach in psychology and educational psychology, introduced by L. S. Vigotsky and his followers in different countries. These chapters are written on the critical and dialectical position and include proposals for creative modification of the present system of education. Some of the chapters contrast positions of Piagetian approach and modern constructivism and cultural-historical approach and base their research on the concepts of orientation of the action, introduction of intellectual activity, differentiation of the goals of the work in preschool and school age, among other proposals. Different ways of understanding of Vygotsky's approach in relation to introduction and teaching of mathematics are reflected in the book.

The book shows theoretical solidity of cultural-historical approach and experience of its implementation in teaching of mathematical knowledge in childhood and the study of the process of psychological development. Of great importance is cross-cultural nature of the whole research, presented in the book, which shows broad possibilities of implementation of cultural-historical theory in the process of organization of mathematical education in different countries as opposition to broadly distributed constructivism. The studies described in the book chapters can be useful for a teacher both in terms of evaluating existing math training programs and in terms of the innovations they can introduce.

The global content of the book can serve as a platform for further unification of the efforts of cultural-historical research and exchange of educational experience in order to obtain better results in the field of teaching and learning of mathematics.

The book is divided in two parts: (1) *Theory and methodology for teaching of mathematics in cultural-historical* approach and (2) *Research and proposals for teaching of mathematics.*

The first part, *Theory and methodology for teaching of mathematics in cultural-historical approach*, includes seven chapters, as follows.

Chapter 2, "*Mathematics as system of cultural-historical knowledge and science*," is written by Mendoza Hernández Octavio (Mexico) and Berstein Erika Mirna (Argentina), researchers from National University of Mexico, UNAM. The chapter analyzes the importance of mathematics as a key piece in the development of scientific knowledge and human thought throughout the time. The mathematics is presented as a science and as an instrument for technological advances. The authors claim that with the help of mathematical models, it is possible to understand and predict the behavior of physical phenomena that are crucial to survive as a species in this complicated universe of which we are a part of. Mathematics, as a science, takes an essential role in the sophistication and technological development of a society. The authors conclude that the correct dissemination and at the appropriate level of understanding of mathematics depends on its appropriate introduction from childhood and from pedagogical work in basic schools.

Chapter 3, entitled as "*Development of symbolic functions and imagination as the basis for learning of mathematic at school: continuation of the dialogue between Piaget and Vygotsky*," is provided by Gonzáles-Moreno Claudia (Pontifica University Javeriana, Bogota, Colombia) Rosas-Rivera, Yolanda (National Pedagogical University, Mexico); Solovieva, Yulia (Autonomous University of Puebla, Autonomous University of Tlaxcala, Mexico and Lomonosov Moscow State University, Russian Federation). The chapter is dedicated to the comparisons of Vygotsky's and Piaget's approach in relation to symbolic development in infancy. The authors present the reflection on the necessity of forming of symbolic function by stages within cultural-historical approach and its development within Piagetian constructivist approach. Different stages (materialized, perceptive, and verbal actions) for formation of symbolic functions and imagination in the learning of mathematics in school-age children are described. Procedures are proposed for the process of calculating and solving mathematical problems from historical-cultural pedagogical psychology based on this relationship. Finally, the role of logical thinking is analyzed considering the way in which logical actions organize the child's activity through external mediation and orientation. The authors conclude that learning mathematics requires the consolidation of the general characteristics of psychological preparation necessary for school and the elements that make up the particular skills related to learning this area.

Chapter 4, "*Psychological means and mathematical concepts acquisition in preschool age: which are better to use?*," is written by Alexander Veraksa and co-authors from Lomonosov Moscow State University. The chapter is dedicated to the process of assessment of the effectiveness of the formation of mathematical concepts of size (length, area, volume) in preschool children aged 6–7 years with different levels of development of regulatory functions. The chapter describes the experiment, in which the concept of magnitude was formed using traditional, traditional with play elements, symbolic, and modeling approaches. The approaches were based on the same actions of children (selection of objects by size through comparison and measurement with the help of measurements), the same methods of their implementation, only the situations in which new concepts were introduced and worked out, and the means offered to children differed. It was found that the development of mathematical abilities during formative classes occurs in students regardless of the approach used. The results of the study outline ways to further investigate the effectiveness of using traditional and non-traditional approaches in the formation of mathematical concepts in preschoolers. The authors discuss their results in Vygotsky-Piaget perspectives on education and educational means in preschool age.

Chapter 5, written by Anastasia Sidneva from Lomonosov Moscow State University, is entitled as "*The pre-number period of teaching mathematics: psychological analysis of the different mathematical programs from the Cultural-Historical point of view*." The objective of this chapter is to discuss some criteria developed in the mainstream of the cultural-historical approach to learning for analyzing the most significant differences in the pre-number period of teaching mathematics from a psychological point of view. The authors show that the most significant differences between the programs concerned the type of concepts proposed for assimilation; the type of actions by which these concepts were to be assimilated and practiced; and how the means of these actions were provided. It seems that the authors of the programs do not always understand what issues these provisions of cultural-historical approach were introduced to resolve. They are, for example, the understanding that the content of instruction and the sequence of studying topics should ensure the rationality of action; eliminate formalism from mathematical concepts; that materialization and modeling are necessary to fix generalized methods for the purpose of further work with these models as means of solving problems, that the speech form is mandatory for the formation of consciousness of action, etc. The chapter concludes that the task of devising a more reflexive and less formal use of cultural-historical approach to create educational programs is still relevant.

Chapter 6 is written by Elaine Sampaio Araujo (University of Sao Paolo, Brazil) and is entitled "*Teaching orienteering activity: principles and practices for the mathematics teaching organization.*" The author has developed the chapter to consideration of OBEDUC Project, which presents pedagogical principles and practices that may set up as basic properties that conduct the mathematical teaching organization at the early years of Elementary Education. The chapter stresses the importance of teaching organization for the results of education. The author claims that the Teaching Orienteering Activity, based on the cultural-historical theory, is a general kind of pedagogical organization and discusses an activity in which this thesis is developed.

Chapter 7, "*Processes of abstraction, generalization, and mathematical concepts formation: a reflection based on the Cultural-Historical Theory*," is written by Josélia Euzébio da Rosa (Universidade do Sul de Santa Catarina—UNISUL, Brazil). The chapter stresses that the teaching of and learning of mathematics constitute one of the challenges for Brazilian school education and the role of positive motivation for learning of mathematics at school and the possibilities of organizing mathematics teaching which trigger the students' learning. Experimental studies were carried out based on dialectical logic and in light of two developments of the Historical-Cultural Theory: Development Teaching Theory and Teaching Guidance Activity. The research involved teachers and students of all the teaching levels. The chapter indicates that the failure in mathematics teaching and learning process depends on the failure of the work with the processes of abstraction, generalization and concepts formation based on logical-formal scheme supported by dialectical logic.

Chapter 8, "Teaching mathematical problem solving in elementary school," is written by Rosas-Rivera, Yolanda, (National Pedagogical University, Mexico), Solovieva, Yulia, (Autonomous University of Puebla, Autonomous University of Tlaxcala, Mexico and Lomonosov Moscow State University, Russian Federation) and Luis Quintanar Rojas, (Autonomous University of Tlaxcala, Mexico). The objective of this chapter is to show a proposal for the analysis of the teaching methods of mathematics that are used in elementary school according to Vygotsky's and Piaget's proposals. In our study, the elements of the teaching method are approached from the theory of activity applied to teaching process. The authors propose qualitative instruments to identify the organization of teachers' teaching methods and the level of assimilation achieved by students. The empirical work, described in the chapter, consisted in two aspects: (1) an interview with third grade teachers, (2) non-participatory observation of classes, and (3) dynamic evaluation of the level of assimilation of mathematical concepts in the students. The results of analysis of the data showed that mathematical content and its organization in different levels of action (materialized, perceptual, and external verbal) are essential for learning mathematics, which are absent in Piagetian and constructivist approach. The authors established profound differences between constructivist Piagetian approach and historical-cultural approach to teaching of mathematics in primary school.

The second part, *Research and proposals for teaching of mathematics*, includes seven chapters, as follows.

Chapter 9, written by Bukhalenkova D. A., Sidneva A. N., Aslanova M. S. and Vasilyeva M. D. (Lomonosov Moscow State university, Moscow, Russia), is entitled "*Development of Mathematical Skills of Primary School Children with Different Levels of Regulation in CHAT-based and Traditional programs.*" This chapter presents the results of a longitudinal study that examined the development of mathematical skills of primary school children in relation to two types of factors: the child's educational environment (more specifically—two programs, one was built according to the principles of CHAT and the other—in more traditional way) and individual cognitive characteristics (children' regulatory skills). This study revealed that it cannot be suggested that one program is more suitable than the other for children with a certain level of executive functions; children with different levels of executive functions develop mathematical skills at their own pace independently of the program they follow. The results of the study are analyzed in two theoretical perspectives of learning and development. The first one, Piaget's approach, emphasizes the need to adapt programs to the child's level of development. The other one, Vygotsky's approach, reflects the idea that training programs should be built in such a way that learning leads to development, and that those mathematical abilities that otherwise would not have arisen, are formed.

Chapter 10, written by Vysotskaya E. V., Yanishevskaya M. A., Lobanova A. D., Taysin M. (Federal Scientific Center for Psychological and Multidisciplinary Research, Moscow, Russia), is entitled "*Scaffolding multiplicative concepts' formation: a way of digital support*." The chapter is dedicated to consideration of balance scale problem as widely known subject for studying the development of thinking and in creating the activity-orientated learning materials. The difficulties of the students during discovery of the general way of achieving equilibrium despite the availability of hands-on trials with appropriate equipment are described. The authors have examined the early stages of multiplicative concepts' formation within the Activity Approach framework and included special modeling means to the inquiry-based learning in order to promote students' actions to the conceptual level of orientation in the coordination of two magnitudes' changes. Different kinds of external orientation and work with the models are presented, so that the students were able to start from the most tricky tasks with several weights located in different positions. The chapter described the principles of digital support design, which aims to scaffold the formation of multiplicative concepts at the very first stages of math education.

Chapter 11, "*The math club as a space for teachers' learning: an analysis based on cultural-historical activity theory*," is elaborated by Brazilian collogues, who work at different universities: Anemari Roesler Luersen Vieira Lopes (Federal University De Santa Maria, Rio Grande Do Sul); Wellington Lima Cedro (Federal Univerisity De Goiás, Goiás). The chapter describes the proposal of the work with the project of The Mathematics Club, which has involved the students from Mathematics and Pedagogy. The project provides a learning space for teaching education once it offers the student teachers a chance to organize, to apply as well as to plan teaching activities. The authors describe the work as based on the idea that one cannot enclose the practice through individualities due to it is something necessarily shared therefore all actions are carried out by everybody involved in the project. The Mathematics Club is developed in a perspective of cultural-historical activity theory. The chapter concludes that, the student teacher searches for conditions that involve him to organize his teaching in order to transform it into his own learning.

Chapter 12, "*The development of mathematical conceptual thinking: a cultural-historical experience in the context of Brazilian public school*," is elaborated by Patrícia Lopes Jorge Franco, Andréa Maturano Longarezi e Fabiana Fiorezi de Marco (Federal University of Uberlandia, Brasil). The authors discuss the development of mathematical conceptual thinking from a historical-cultural experience in the context of the Brazilian public school. The study was based on the dialectical-historical materialist method and the Teaching Guiding Activity, whose theoretical-methodological guidelines favor subjects to interrelate their teaching and study activities through a didactic-formative intervention procedure. The chapter discusses how the organization of teaching for the development of students' mathematical conceptual thinking is constituted as a content-form of development of those involved. The results of the study indicate that the teacher presented the development of mental action on teaching, offering didactic elements capable of guiding students to think and act through scientific concepts in Algebra. The authors concluded that the students achieved a certain mastery of the means and of the logical mental actions of the internal movement represented by the theoretical concept of functions and linear and quadratic equations, based on the links and nexuses that compose them.

Chapter 13, "*Contributions of the cultural-historical theory in the teaching activity of time and area measures*," is written by Ana Paula Gladcheff Munhoz (University of São Paulo); Moisés Alves Fraga (University of São Paulo); Neusa Maria Marques de Souza UFMS, (Instituto de Matemática e Estatística, Post-graduate program in Mathematics); Amanda Cristina Tedesco Piovezan (University of São Paulo); Anelisa Kisielewski Esteves (University Anhanguera Uniderp, Master's Program in Science and Mathematics). The chapter is dedicated to the topic of the development of the measurement concept at school depends directly on determining the quality to be quantified. The authors provide three essential phases of the process of appropriation of knowledge: highlighting the quantities to be measured; comparing quantities of the same nature and numerically expressing the result of the comparison. This chapter approaches the work with quantities that, perhaps, are considered the most complex to be understood, mainly by children: the quantities time and area. The authors provide the evidence of elaboration and discussion of teaching situations for these concepts, revealing and quantifying the quality that they express, based on cultural-historical theory and on the assumptions of the teaching-orienteering activity. The authors discuss a "non-procedural" teaching, which transcends the reading of hours and calendars, the application of formulas disconnected from meanings and dialogue about possibilities for the development of work with measurements in the early years of Elementary School, through Learning Triggering Situations.

Chapter 14 is written by Inger Eriksson from Sweden and entitled as "*Collective reasoning and the use of learning models for relationships between quantities, as suggested by the El'konin Davydov curriculum.*" This chapter focuses on learning activity that was suggested by El'konin and Davydov as a framework for teachers to design opportunities for students to emerge collective mathematical reasoning. The data for the chapter were taken from a 3-year longitudinal research project in which learning activity was used to design teaching, tasks, and learning models in a multicultural and multilingual Swedish primary school. As a result of using this framework, a situation was highlighted whereby the learning models specific to learning activity had been constructed by second language students aged 6 and 7 years as they emerged collective mathematical reasoning that focused on comparing two different quantities. The author points out the contribution to previous research concerning how learning models can be used in such student groups for collective reasoning about relationships between quantities is twofold; that is, (1) the use of the learning models can help students and teachers to return to the problem they have jointly identified in the collective reasoning and (2) the use of the learning models can help students who struggle with different linguistic dilemmas to jointly reflect on both their own and others' arguments.

Chapter 15, written by the collective of the authors from Sweden, Inger Eriksson, Jenny Fred, Anna-Karin Nordin, Martin Nyman & Sanna Wettergren, is entitled "*Learning models" as a means of materialising algebraic thinking in joint actions—results from a design study in grades 1 and 5 in Sweden.*" The chapter is based on a design research project based on Davydov's principles of learning activity. The aim of the chapter is to exemplify the functions learning models can have in visualizing students' algebraic thinking when they collectively discuss algebraic expressions. The data was comprised of videotaped research lessons in grades 1 and 5 of primary school, respectively. The analysis indicates that the learning models enhanced the students' joint exploration of the idea behind algebraic expressions and thus materialized their algebraic thinking in the whole-class discussions in three ways: (1) materializing an argument, (2) materializing a problem, and (3) materializing a collective memory.

The book is useful for broad sector of the readers, such as pedagogues, psychologists, educators, neuropsychologist, and all professionals interested in the topic of conceptual development of the children and organization of the process of teaching and learning of mathematics.

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Part I
Theory and Methodology for Teaching of Mathematics in Cultural-Historical Approach

Chapter 2
Mathematics as a System of Cultural Historical Knowledge and Science

Mendoza Hernández Octavio

In this chapter, we analyze the importance of mathematics as a key piece in the development of scientific knowledge and human thought throughout the time.

Thanks to mathematics, many of the technological advances that we enjoy today have been materialized. In addition, with the help of mathematical models, it is possible for us to understand and predict the behavior of physical phenomena that are crucial to survive as a species in this complicated universe of which we are a part of.

The sophistication and technological development of a society, in the moment of time when you look at it, is a true reflection of the complexity of the mathematics which are available. Therefore, the correct dissemination and at the appropriate level of it, from childhood and in basic schools, is essential for this process of progress in the future to be maintained and bear fruit.

The development of abstraction processes in human thought is a rather complex phenomenon. The formal thinking (which allows us to “calculate”) and the so-called intuitive thinking (which allows us to create) must be encouraged and developed from childhood if we want to maintain our scientific advances and not fall into backwardness and darkness, as has already happened in our history. It is important to note that the society of our days could not work without mathematics. All technology depends on it and is often present without noticing it. Sometimes it is used mathematics that have been developed two thousand years ago, other times we use more recent mathematics, and the above is due to the fact that mathematical discoveries never expire and rather they fructify and interconnect with the new developments in a very surprising ways.

The development of mathematics has accompanied, since ancient times, to the growth of humanity in all aspects. It is believed that the development of the incipient

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mathematics of the early days, motivated by the resolution of everyday problems, has preceded and promoted writing. The first known records of this phenomenon date from the time of the Babylonians more than 10,000 years ago.

The humanity, in its long journey through time, has gone through several fundamental moments where miraculous leaps have been made in mathematical development, which have given a transcendental boost to the tasks of scientific knowledge, and also to everyday things such as trade, agriculture, medicine, and navigation.

In what follows, and in order not to lengthen or complicate the discourse, we will give a summary outline of some of these periods. We recommend to the readers, who want to know in more depth the history of mathematics throughout the centuries, the works of Steward (2014) and Wussing (1998).

2.1 The First Leap Toward Abstraction in Human Thought: The Concept of Number

The concept of number is one of the first miracles that we have had in our long development since the most remote times. It is an extremely complex concept that became real and visible as a result of several very fortunate events that led us to it. Paraphrasing Steward (2014), according to known records "everything started with small clay tokens, 10,000 years ago in the Near East". According to Steward, more than 10,000 years ago, in the peoples of Mesopotamia, the accountants of those times already kept records of the assets of each owner, even if writing had not yet been invented and symbols for numbers did not exist. Instead of numbers (because they did not yet exist) small clay tokens of different shapes were used, with which they represented basic products of the time. For example, the spherical clay tokens represented bushels of grain, the cylindrical ones represented animals (cattle), and the ovoid ones represented jugs of oil. As time went by, the tokens became more and more complicated to represent more goods. This way of representing and counting everyday goods is the first fundamental step on the path toward number symbols and the concept of number. According to Steward's narrations, these tokens were used to keep records of assets, for tax or financial purposes, or as legal proof of ownership. To avoid falsifications in these property records, the accountants of those times kept said tokens (which corresponded to the assets of a certain person) in a clay container which they sealed to prevent fraud, and when an inventory was required, it was broken the vessel that contained the tokens, in order to know what the owner had. However, repeatedly breaking the container to learn of the contents or to update it was a cumbersome task. So it was decided to draw symbols on the outside of the vessel that told us about the content. At some point they realized that it was no longer needed the clay tokens of the vessels since the symbols used to describe the content were more than enough. This obvious but crucial step is the starting point of what was the writing and the symbols to represent the numerals. The marks on these vessels evolved little by little until they became the numerals we know today:

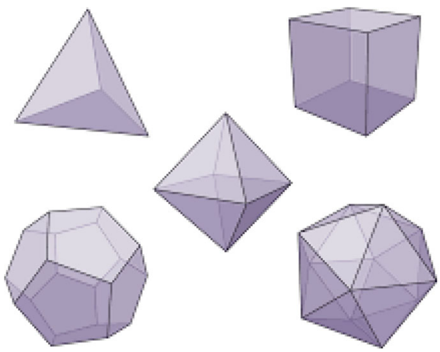
0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. However, it should be noted that the previous numerals and the abstract concept of number took the humanity several thousand more years to come to fruition. What prompted such a development? Well, with the passing of the millennia and the development of agriculture, permanent settlements of people appeared, forming a series of city states: Babylon, Erido, Lagash, Sumer, and Ur. With the impulse of such development, it is believed that the primitive symbols inscribed on tablets gave rise to the first writing, which is recorded, called "cuneiform" and that was developed by the Sumerians around the year 3000 BC.

For those times, the Sumerians already used a numbering system that they applied for trade and daily accounting and also in astronomy. The numeral system invented by the Sumerians was positional with base 60, and there are still certain vestiges of use nowadays since is used to measure time (1 h is 60 min, and 1 min is 60 s) and angular measurements (a complete turnaround the circle is 360 degrees). The Sumerian sexagesimal numeral system did not use a sign for zero, which did not appear until Hellenistic times (323-31 B.C.). From the mathematical point of view, some of the most important novelties, recorded in the Babylonian tablets, refer to the solution of linear and quadratic equations and also to the so-called "Pythagorean theorem" and its numerical consequences, which was enunciated in all its generality and proved in the famous book "The Elements" by Euclid that appeared until the Hellenistic period.

2.2 The Beginnings of Geometry and the Axiomatic Method

It can be said that the symbolic language of mathematics began in the Mesopotamian period with the birth of the idea of number and its different representations. This notion was very important since allowed a type of reasoning called "symbolic" which consisted of the manipulation of symbols (numbers, variables, or unknowns) to solve some of the first accounting problems of humanity. It should be noted that, starting from the middle ages, this symbolic reasoning took on a new impulse and eventually led us to the invention of algebra and mathematical logic, which have played a fundamental role in the development of modern mathematics of nowadays.

The interest in shapes, images, and drawings are part of what we could call "visual" reasoning that is of great help in understanding certain abstract concepts of mathematics, both ancient and modern. Although images are less formal than symbols, it should be noted that they are a powerful and potential incentive for abstract ideas and the development of what could be called "mathematical intuition", without which a mathematician of our days is incapable of promoting the development of it. Such a visual intuition is a fundamental characteristic in order to introduce and develop the study of shapes, that is, geometry. Moreover, it is somehow a measure that shows us the development and complexity of the human brain. Although the interest of the ancients in shapes already appears in some Babylonian tablets, the first rigorous use of shapes (geometry) along with the introduction of certain formal rules to manipulate them occur in the work "The Elements" by Euclid of Alexandria, also known as the "father of geometry", who is believed to have lived between 325 and 265 B.C. It should be noted that, in his work, Euclid highlights the importance that any geometric statement, which is supposed to be true in the eyes of "geometric intuition", must have a rigorous logical proof. The highlight point of Euclid's work is the proof that there are exactly five regular solids (made up of identical faces): the tetrahedron (made up of 4 equilateral triangles), the cube (made up of 6 squares), the octahedron (made up of eight equilateral triangles), the dodecahedron (formed by twelve regular pentagons), and the icosahedron (formed by 20 equilateral triangles). Below we show a figure of the five regular solids, also known as "platonic solids".



The importance of Euclid's work "The Elements" is that, in order to synthesize and to structure the knowledge of his time, Euclid invented a procedure called the "axiomatic method" with which he was able to achieve his goal in a brilliant and spectacular way. Such axiomatic procedure is of crucial importance nowadays for the development and systematization of the mathematical knowledge. To give an idea of the importance of the axiomatic method, Euclid did not limit himself to affirming that a theorem is true, but also offered a rigorous proof of it, unlike his predecessors where only this or that result was stated as if it were a recipe for kitchen room. However, what is a theorem? and what is a proof? A theorem in geometry is a statement that tells us about certain properties that a geometric figure is supposed to satisfy (e.g., the sum of the internal angles of a triangle gives us 180°), which does not have to be evident. A proof of a theorem is a finite sequence of steps, where each step is a logical consequence of the previous one, until reaching the statement that the theorem tells us. Now, each statement that is affirmed, in the steps of the proof of the theorem, has to be justified by referring to previous theorems. At this point, the greatness of Euclid is that he realized that this chain of reasoning cannot be carried backwards indefinitely and that there must therefore be a finite number of statements that cannot be proved and with which (as if were bricks) it was possible to build the "building" of geometry. In order to do that, Euclid started with: (a) a list of primitive notions (which would play a role similar to our alphabet), for example: point, line, circle, triangle, angle, etc., with which to build geometric statements, and (b) a list of "axioms" (known as Euclid's postulates), which are geometric statements assumed to be true because they are obvious (this is where geometric intuition is important). These postulates are the following:

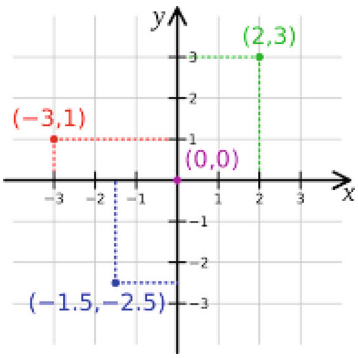
(P1) By two distinct points only crosses a straight line.
(P2) A segment can be extended in an unlimited line.
(P3) A circle can be drawn from a central point and any radius.
(P4) All the right angles are equal to each other.
sum is less than two right angles, then those two lines will intersect on that side.

According to Euclid, the geometry is reduced to a logical game (like chess), where theorems are built from primitive notions and axioms, using the logic. In analogy with chess, the primitive notions are the chess pieces and the board; and the axioms or postulates are the rules of the game. In this case, a theorem is something like a determined position (at some point in the game) of the chess pieces, and the proof of such theorem is precisely the process (game moves) that led us to such position. Despite its purely playful aspect, the geometry has had a major impact on the development of physics and chemistry. One of the most surprising applications, made since ancient times, was given by Eratosthenes of Cyrene, who around 250 B.C. managed to estimate the size of the earth. According to Eratosthenes computation, the circumference of the earth is 39,250 km, and the current one is 39,840 km.

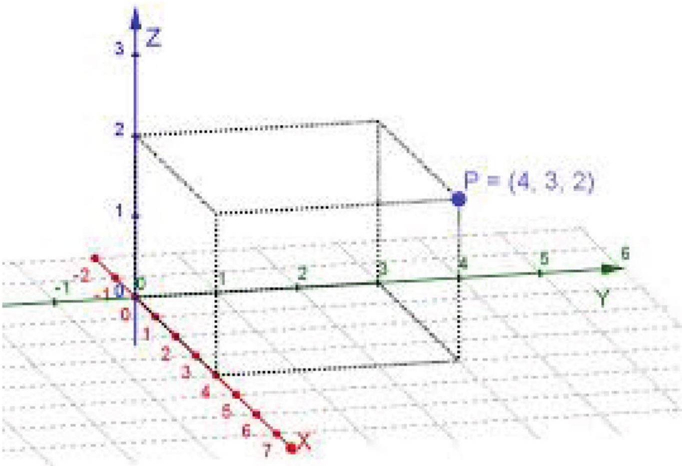
2.3 The Method of Coordinates and the Infinitesimal Calculus

For the purposes of human convenience, and to simplify its dissemination and teaching, mathematics is usually divided or classified into areas such as: geometry, algebra, number theory, topology, and analysis. On the other hand, modern mathematical thought teaches us that there are no rigid boundaries between apparently different areas. Rather, there are connecting bridges between areas that are supposed to be foreign, through which the problems from one area can be solved using methods from another. The greatest advances in mathematics research often occur when an unexpected connection is made between areas that previously seemed completely disconnected.

One of the first surprising connections between areas other than mathematics was established by René Descartes (1596−1650), who managed to establish a connecting bridge between algebra and geometry using his method of coordinates. In everyday life we are used to spaces of two and three dimensions (the plane and the space). The key to interpreting multidimensional spaces is precisely to introduce what is currently called a coordinate system. The brilliant idea that Descartes had was to interpret the geometry of the plane by assigning to each point P of the plane a pair of real numbers (x, y) called coordinates, in the following way: a point O of the plane (which we will call the origin) is chosen; then, through the origin we draw two lines (coordinate axes) that intersect at the origin point forming right angles. With this, we have the west–east axis (horizontal) and the south–north axis (vertical). Then, the coordinates (x, y) give us the position of point P, with respect to these coordinate axes, and can be interpreted as follows. We start at the point O and move a distance x (respectively, $-x$) to the east (west) if x is positive (respectively, negative); and then a distance y (respectively, $-y$) to the north (south) if y is positive (respectively, negative), thus arriving at the point P. In particular, note that the origin point O has as coordinates the pair (0, 0). Other examples of coordinates of points in the plane can be seen in the following figure.



Coordinates work similarly in the space, and only now we need three coordinates (x, y, z), as can be illustrated in the following figure.



Following the previous idea of the coordinate axes, the number of dimensions of the geometric space in question is determined by the quantity of numbers that we need to specify the position of a given point with respect to the chosen origin. In this way the “geometric loci” and “graphs” (points, lines, curves, etc.) can be interpreted and studied by using algebraic expressions. For example, in the case of the plane, each equation in two variables of the form $ax + by + c = 0$, with $a^2 + b^2$ not zero, corresponds exactly to a line in the plane and vice versa. On the other hand, a circle of radius r and centered at the point with coordinates (a, b) corresponds to the algebraic expression $(x - a)^2 + (y - b)^2 = r^2$.

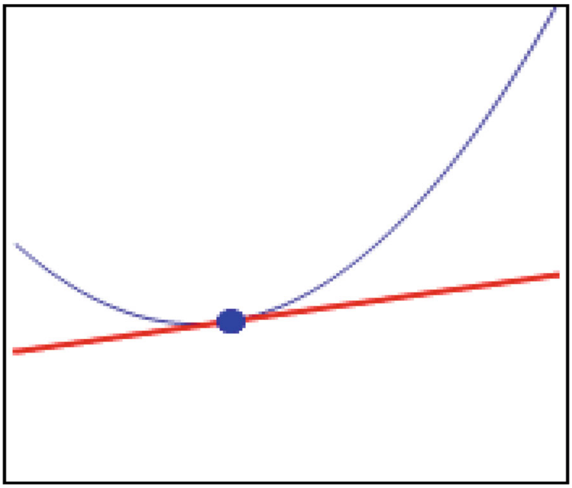
It is worth mentioning that the coordinate method was also developed by Pierre de Fermat (1607−1665), who worked on the same type of problems as Descartes did at approximately the same time. The works of Descartes and Fermat complemented each other, and this is why both are considered the fathers of what is known today as “analytical geometry”, which will become a key piece for the born of the so-called infinitesimal calculus.

The coordinate method has been fundamental for the development of mathematics and other sciences. Such a method is used in everyday situations, and we are not even aware of it. For example, all the computer graphics use an internal coordinate system, and the geometry that appears on our screens is handled algebraically. Coordinate systems are used for navigation systems, both for ships and aircraft, and for mapping. The effectiveness of mathematics, as a way of understanding and modeling the universe around us, is due to the fact that it can be adapted and used to transfer ideas from one area of science to another. Therefore, mathematics is an indispensable tool for technology transfer and the development of science in general.

In the middle of the sixteen century and at the end of the seventeenth, the transition from feudalism to capitalism took place. This brought, among other things, the consolidation of mathematics and natural sciences in society with an intensity that grew as the new production regime was consolidated. The mathematics is nourished more and more by the technical problems arising from the development of the productive forces and the natural sciences. With this, it became crucial to develop a mathematical theory that would serve as a tool to understand the movement of bodies and that would allow the development of practical mechanisms and tools for the industrial development. Hydraulic engineers and builders (of ships and fortifications) needed a more theoretical treatment of the physics–mathematics sciences (which did not exist), with which they could face the enormous number of technical problems they faced in their day to day. For example, the energy relations between the moving parts of machinery (hydraulic wheels, etc.) were carried out with empirical and approximate calculations since the mathematics of that time was not sufficiently developed or simply did not exist. The infinitesimal calculus was the mathematical theory, invented in 1680, which served as the basis to solve all these shortcomings and to develop technical–scientific progress to unsuspected levels. This work was possible thanks to the previous works of Pierre de Fermat and René Descartes. The fathers of the calculus were Isaac Newton (1643−1727) and Gottfried Wilhelm Leibniz (1646−1716). It should be noted that before the invention of infinitesimal calculus there were a large number of failed attempts, which served as a basis to better understand the concept of "variable quantity" and that of "instantaneous change" of a given magnitude.

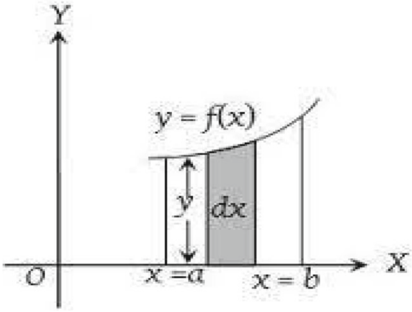
The infinitesimal calculus has as its main idea to describe and study the process of "instantaneous change" of a specific magnitude (velocity, population growth, temperature, pressure, etc.) and has two main branches: (1) the differential calculus, and (2) the integral calculus. Given a curve in the plane, the "geometric interpretation" of the differential calculus has to do with the problem of finding the tangent line to a point on such curve and that of the integral calculus is related with the problem of finding the area under such curve.

The tangent line to a given point on the curve:



The area under the curve given by the expression $y = f(x)$ and included between the lines $x = a$ and $x = b$:

$$\text{Area} = \int_a^b y\,dx = \int_a^b f(x)dx$$



The infinitesimal calculus became a powerful tool that served both physicists, chemists, and engineers to obtain mathematical models (through the so-called differential equations) of certain natural systems. For example, in the design of airplanes, automobiles, suspension bridges, buildings, growth of a given population, the behavior of epidemics, the calculation of trajectories of space probes, etc.

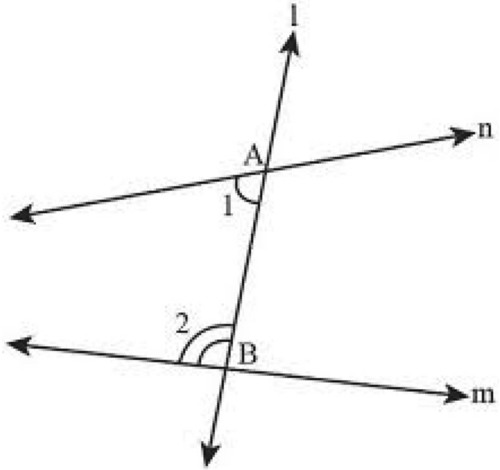
The impetuous development of the natural sciences, supported by the new tools provided by the infinitesimal calculus (and its subsequent developments), resulted in the emergence of new technical sciences and engineering. These new sciences had to be organized with the creation of new scientific centers called “polytechnic schools”. The main center, which served as the base for those that followed, was the very famous “Ecole Polytechnique” in Paris, founded in 1794.

With the industrial revolution and the consolidation of the means of production, mathematics was occupying a central place in the training of engineers in higher technical schools. Furthermore, together with the natural sciences, they began to constitute solid specialties that founded independent scientific and mathematical institutes, where mathematics is one of the fundamental subjects.

2.4 The Non-Euclidean Geometries

At the end of the nineteen century, the geometry flourished with a vertiginous development of its scope, contents, new methods, and internal structuring. One of the starting points, in this new development, has to do with the famous “Euclid’s fifth postulate” (P5) which says that: If a line l cuts two others n and m forming interior angles on one side, provided that their sum is less than two right angles, then those two lines will intersect on that side.

In the following figure, we illustrate (P5), where it can be seen that lines n and m intersect exactly on the side where the sum of angles 1 and 2 are less than two right angles:



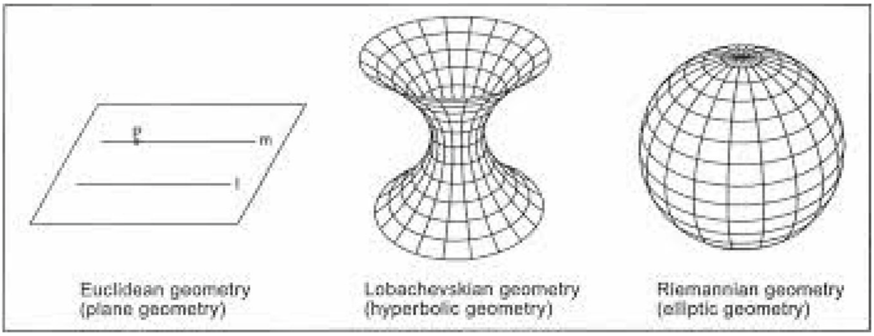
This postulate was not evident to many mathematicians, and therefore, they wanted to deduce it as a theorem from the other four. Despite all the frustrated attempts, at the end of the eighteen century, other hypotheses equivalent to the fifth postulate had already been found, such as: (q5) the sum of the internal angles of a triangle is equal to two right angles, or also (p5) given a line L and a point P that is not on such line, there exists a unique line M which passes through P and is parallel to L. Note that, although (p5) is somewhat evident, in the case of (q5) is no longer so; and yet they are equivalent to the fifth postulate (P5).

The continued failures to prove Euclid's fifth postulate led to a change of approach to this problem. Instead of proving this postulate directly, the so-called proof method by "reduction to absurdity" was tried. That is, replace the fifth postulate with a new one that consists of the negation of (p5), and with the help of the other four, obtain some contradiction. Such a scheme of indirect proof of the fifth postulate continued unsuccessfully well into the nineteen century.

The first mathematician who understood that the fifth postulate could not be deduced from the other four and that therefore other perfectly valid geometries could be obtained was Carl Friedrich Gauss (1777−1855). This leap in thought caused an earthquake in the scientific world, since he raised the need to differentiate between geometry (as a mathematical discipline) and space (as a physical science of our world). On the real structure of our physical space, physics and astronomy had to decide, as really happened some time later, with the famous "general theory of relativity" of the physicist Albert Einstein (1879−1955), where the mathematical apparatus necessary to Einstein's theory of relativity was supported precisely by these new geometries developed by the mathematicians Janos Bolyai

(1802–1860), Nikolai Ivanovich Lobachevski (1792–1856), and Georg Friedrich Bernhard Riemann (1826–1866).

The non-Euclidean geometries that Bolyai, Gauss, and Lobachevski developed are currently known as "hyperbolic geometries"; in these geometries it can be shown that for a line L and a point P that is not in L there are infinite lines parallel to L passing through point P; in hyperbolic geometries, the sum of the interior angles of a triangle is less than two right angles. On the other hand, Riemann developed the so-called "elliptical geometries" in which it is proved that for a straight line L and a point P that is not in L, there is no straight line parallel to L that passes through the point P; in elliptical geometries, the sum of the interior angles of a triangle is greater than two right angles. It is worth mentioning that it was precisely Riemann who raised the problem of the study of space in the most general way possible, considering any n-dimensional variety for it. This mathematical work by Riemann, on n-dimensional geometries and space, was the basis on which Einstein's general theory of relativity was founded.



In summary, we can say that the change of thought between geometry and space, given by mathematicians, caused a revolution in the world of physics, making possible many of the technological advances that we now enjoy without realizing it. For example, the famous GPS navigation system is made possible by Einstein's general theory of relativity. This GPS system is essential in all the transportation systems in the world; emergency services and disaster relief also rely on GPS. Many daily activities such as banking, mobile telephony, and electricity distribution networks are more efficient thanks to the chronometric accuracy provided by GPS.

2.5 The Mathematical Logic and the Set Theory

At the beginning of the twentie century, the path of the axiomatization of geometry reached its culmination with the work of the mathematician David Hilbert (1862–1943). Hilbert discovered flaws and logical errors in the system of axioms posed by

Euclid. To remedy them, he posed a new list of axioms (21 in all) and discussed his role in the Euclidean geometry.

Hilbert also raised the requirements that a system of axioms must meet in the construction of any mathematical theory. The first requirement of such a theory is that it be consistent, which means that the axioms are not contradictory; and the second one is that such system of axioms be complete; that is, to be broad enough to be able to demonstrate any theorem that appears in the content of such theory. The proof that non-Euclidean geometries are possible shows that the Euclid's fifth postulate is independent of the other four.

In his work on the axiomatization of geometry, Hilbert showed that the axioms of geometry can be reduced to those of arithmetic, and therefore, the consistency of geometry reduces to the consistency of arithmetic. However, this reduction did not solve the problem, since at that time it was not known if the arithmetic was consistent or not; and even more, it was not even clear what are the natural numbers. With respect to the natural numbers, Giuseppe Peano (1858−1932) raised in 1889 a series of three axioms that described the main properties of such numbers:

(A1) There exists the number 0;
(A2) Every natural number n has a successor s(n);
(A3) (Principle of mathematical induction) If $P(n)$ is a property of numbers such that: $P(0)$ is true, and if every time $P(n)$ is true, it follows that $P(s(n))$ is also true, and then $P(n)$ is true for all natural n.

Using the above axiomatics, Peano was able to define the natural numbers 1, 2, 3,..., in terms of such axioms. For example, $1 = s(0)$, $2 = s(s(0))$, and so on. He also defined the operations of addition and product of natural numbers and showed that such a system satisfies the usual properties. One of the advantages of this axiomatization is that, if we want to prove that the natural numbers exist, we have to construct a system that satisfies all the Peano's axioms. Therefore, the construction of such a system became a fundamental problem at that time. In 1884, Gottlob Frege (1848−1925) tried to solve this problem by building the natural numbers from simpler objects: sets. With such a construction, Frege discovered that he could reduce all arithmetic of natural numbers to logic. One of the assumptions that Frege had made, in his foundation of the natural numbers, is that any "reasonable" property legitimately defined a set which consisted of all the elements satisfying that property. Unfortunately, Bertrand Arthur Willian Russell (1872−1970) showed with an example (now known as the Russell's paradox) that not any "reasonable" property could define a set. Russell tried to repair the flaw in Fegre's construction. For this he had to restrict the type of properties that could be used to define a set and show that such restrictions did not lead to paradoxes. In the period from 1910 to 1913, Russell and Alfred North Whitehead (1861−1947) in their work "Principia Mathematica" created a complicated technique called "type theory" that achieved this goal.

In 1920, Hilbert proposed a research project, which became known as the "Hilbert Program", in which it was intended that mathematics be formulated on a solid and logical basis. Hilbert believed that all mathematics followed from a finite system of correctly chosen axioms, and that such an axiomatic system could be proved to be consistent. He also believed that every mathematical statement could be proved or disproved and that, in principle, every mathematical problem could be solved. Unfortunately, in 1931, Kurt Gödel (1906−1978) tore Hilbert's program to pieces. In that year, Gödel showed that it cannot be proved the completeness of any large enough non-contradictory formal system containing Peano's axiomatics. Therefore, there exist mathematical statements that cannot be proved or disproved. An intriguing consequence of Gödel's work is that any axiomatic system for mathematics must be incomplete: it is not possible to elaborate a complete and non-contradictory system of axioms that englobe the development of the mathematics. Gödel's work resulted in the development of the computability theory and later the mathematical logic as an autonomous discipline in the 1930s−1940s. All of the above served as the basis for what would be the theoretical computing developed by Alonso Church (1903−1995) and Alan Turing (1912−1954), which has had an impressive impact on today's technology, making possible the advent of the computers.

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Chapter 3
Development of Symbolic Functions as the Basis for Learning of Mathematics at School: Continuation of the Dialogue Between Piaget and Vygotsky

Claudia Ximena González-Moreno, Yolanda Rosas-Rivera, and Yulia Solovieva

Abstract This chapter compares the approaches of Vygotsky and Piaget in relation to the development of the symbolic function for learning mathematics. In the historical–cultural approach proposed by Vygotsky, the topic of acquisition of symbolic means is one of the central aspects of development in infancy. The use of symbolic means should become reflexive at the end of preschool age, as one of the indicators of readiness for school. The chapter considers the concept of orientation and gradual formation of symbolic functions on different levels as the principal differences between these approaches. The chapter analyzes the levels of formation of symbolic function as the level materialized actions, perceptive actions, and verbal actions with external mediatization and with the orientation of the adult. The analysis is provided on the basis of consideration of the practice of formation of symbolic function in the reality of child's development. The chapter concludes that for Vygotsky's followers, this concept became the key position for the whole development, including mathematical thinking.

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3.1 Introduction

In the historical–cultural approach proposed by Vygotsky, the topic of acquisition of symbolic means is one of the central aspects of development in infancy (Salmina, 2019). The symbolic function constitutes an essential element of all kinds of intellectual activity, directed to solution of established concrete goals. Intellectual activity passes through different levels of development, started from the level of material actions, materialized actions, perceptive, and verbal oral and written actions (Galperin, 2000; Solovieva, 2022). Both external intellectual activity and internal thinking process occur not in direct spontaneous, but on the basis of symbolic means, acquired in different activities during one's practical and intellectual activity. Mental actions can be determined as the ability to carry out a certain transformation of the object, in the mental plan of a subject (Galperin, 1986, 2000) and, therefore, transform intellectual activity into internal process of thinking. Intellectual actions always configure the levels of development of the symbolic function.

The symbolic function implies the conscious and reflective use of various signs and symbols (Salmina, 2019). Diverse cultural sings and symbols are used by a subject reflexively in the actions of representation. This representation consists of expressing the same content in different ways (Lotman, 2001). The symbolic function implies the conscious and reflective use of various signs and symbols. Reflexive use means that the child understands that it is about signs and symbols; that is, it is about representation of something by means of something else. Developed representation consists of expressing the same content in different possible ways.

It is necessary to stress that reflexive use of the symbols and signs mean that the child understands that it is about; that the signs and symbols are substitutes on something else and that are the other possibilities of representing something by means of something. It is possible to suppose that symbolic function does not exist unconsciously, that the subject who is representing something, knows that he or she is representing. It is not possible to represent something without understanding of the meaning of the chosen mean for representation. Symbolic function without its conscious reflection converts in simple imitation, which is characteristic of initial preschooler. The symbolic function allows the child to make symbolic representations of reality that go beyond experience or direct contact with the environment.

The final period of preschool age, approximately starting from 5 years, represents the period, when the variety of symbolic means might be not only used, but also created by the child (Gonzalez-Moreno, 2016; Luria, 2016). This ability is not achieved spontaneously, only by having the age of 5, but is the result of specifically organized activity with proper content in appropriate conditions. Specific kinds of activity are necessary to guarantee and enrich development and perfection of symbolic function. In different psychological studies, it was shown that the symbolic function makes it possible to create mental images through actions with objects (object substitutes and drawings) and imaginative actions (Gonzalez-Moreno & Solovieva, 2016).

Psychological research affirms that it is necessary to anticipate the use of the symbolic function and guarantee the conditions in which its use is conscious and voluntary for the child. The ideal situation is when the child themselves, along with other children, know, use, modify and propose various symbolic media, and understand their meanings. The appropriate activity to achieve this objective is, undoubtedly, children's play. It is not about the play free and simple that is reduced to the manipulation of objects, and this is a collective activity directed to a purpose representative: thematic social role-play (Elkonin, 1989).

The content of the thematic social role-play is the activity that allows introducing various symbolic means, from substitution to the schematization of behavior in the game under certain rules, mutually agreed and understood (Solovieva & Quintanar, 2019). During this activity, it is possible to introduce the symbolic means as simple substitutions of material objects and practical actions with them. Later, during organization of thematic plays with social roles and variety of different rules, it possibly raises their level of complexity, from the materialized and perceptual level to the verbal level in oral speech (Gonzalez-Moreno & Solovieva, 2016). It is also possible to increase the initiative of its use in older preschool children during collective activities, for the development of imagination, as another essential neoformation of the preschool age. Specific organization of the creation, reflection, and continuous use of diversity of symbolic means in narrative plays may be useful for this purpose (Gonzalez-Moreno, 2021).

Representatives of activity theory applied to ontogenetic development have shown that the level of development of the symbolic function forms an important indicator of the child's psychological preparation for school (Salmina, 2019). Such indicator might be evaluated during the period of the transition from preschool to school age. The objective of this chapter is to show the relevance of symbolic function and its step-by-step formation in orientated collective activities.

3.2 Symbolic Functions and Mathematics as Cultural Intellectual Actions

Symbolic function is essential element of initial preparation of the child for school learning in general and, especially, for learning of mathematics as a scientific matter. Learning mathematics makes it possible to think logically, observe, establish relationships, find, and represent patterns (shapes, regularities, and structures), make assumptions, and see what happens (what is a number, what logical perspective one has on the world, and reason from the assumptions) in a particular language (Antonsen, 2021). This particular language is the language of mathematical means, which are based on logic and symbolism. The use of mathematical language means that someone may generalize and convert in abstractions some features of the phenomenon of the real world, express them in symbolic manner, and realize diverse operation with them.

Such capacity in not innate but might be formed during infancy by inclusion of the child into specific kinds of activity, which require the use of substations of objects, actions, rules, and algorithms of communication. Acquisition of mathematical concepts of school age should be based on the previous development of symbolic functions in preschool period (Salmina, 2019). The teaching of mathematics requires a precise methodology for the formation of the concept of number, the decimal number system, and the development of mathematical actions such as measurement of different magnitudes, comparison of the magnitudes, and numerical conversion of different magnitude measures by diversity of means (Rosas et al., 2019). All these complex intellectual actions take place not with the objects itself, but with generalized features of the objects, such as magnitude, area, and volume (Veraksa et al., 2022). All these actions imply the use not of mechanic imitation, but of the conscious and reflective use of various signs and symbols (Solovieva et al., 2021). Such use of symbolic function in mathematic is not possible without gradual formation by using various forms of representation of the relationships between objects, images, and actions (Gonzalez-Moreno & Solovieva, 2022; Solovieva & Gonzalez-Moreno, 2017).

Learning mathematics at school age enables the intellectual development of children by allowing them to use analytical thinking, fulfill logical operations on different levels of intellectual actions, to improve decision-making, and to come to solutions quickly and correctly. The complexity and conceptual richness involved in the teaching of mathematics is evident. However, the current programs at the preschool and initial primary level do not consider this complexity and are limited to a direct, empirical, mechanic execution of certain operations, among which are undoubtedly included those established in the works of Piaget (1975), based on the supposition that reflection and logic will appear till the age of twelve.

The main point of discussion is: do these abilities appear spontaneously according to the child's age or they might be introduced and formed in children gradually, according to the conception of step-by-step formation of the action (Galperin, 2000; Solovieva, 2022).

Constructivism approach and cultural historical conception of development answer differently to this question.

3.3 Piagetian Perspective of the Formation of the Symbolic Function

According to constructivism, intellectual abilities appear spontaneously and somehow create in children a conscious disposition to undertake actions that lead to the solution of the problems they face every day in a coherent, orderly, and effective way because it enhances the reasoning capacity. According to constructivism, mathematics is included in all daily actions, which is why success in learning mathematics is related to success in life. For example, with mathematics savings can be managed, time can be managed, and it also generates a sharp abstraction capacity to continue learning throughout life and makes it possible to mathematically represent real world situations. It is possible to disagree with this inclusive position, saying that children suffer a lot in primary school, while they simply understand nothing from mechanic operations they are asked to fulfill by the teachers. To kind of day-to-day practice and life help them to understand mathematics and to pass from empirical counting to abstract concepts.

According to Piaget (1975), the process of adaptation in the domain of cognition (that is, the construction of knowledge) is a process of conceptual equilibrium that takes place within the knowing organism, while the development of cognition implies the expansion of a balance that leads to cognitive change.

Piaget (1959) believed that "knowledge is a higher form of adaptation" (p. 12). Consequently, Piaget viewed cognition as an adaptive tool for the construction of a viable conceptual structure. To build the foundation of his theory of learning and development, Piaget used the interrelated concepts of assimilation and accommodation. Assimilation is how human beings perceive and adapt to new information (Piaget, 1975). It is the process of fitting new information into pre-existing cognitive schemata (Piaget & Inhelder, 1973). Accommodation is the process of the individual taking in new information from the environment and upsetting pre-existing thought patterns to accommodate the new information (Piaget, 1975).

It is curious to remember that the symbolic function was approached by Piaget from his theory of intellectual development. According to Piaget (1959), intellectual development considers three lines of development: (1) operational, which corresponds to development by age and characterizes qualitative changes, (2) acquisition of specific-object knowledge, which refers to functional development, this means the assimilation of different means of actions, obtaining more content but without qualitative changes, and (3) acquisition of semantic knowledge.

The method of evaluation of development created by Piaget (1959) is based on the analysis of a logical development of the child. Supposedly, such development is spontaneous. During the testing, the child responses to the tasks for conservation, seriation, and correspondence. All these tasks require the child's possibility for generalization of some external properties of the objects and their representation by symbolic means. The responses to the tasks proposed in this method allow to establish the level of development of the symbolic function that has already been reached by the child but does not allow us to understand how this level has been reached. Piaget considered the interview necessary to identify the mental processes through which the child constructs knowledge, as well as to relate it to the repercussions in the classroom (Kato & Kamii, 2001).

Piaget made contributions to the understanding of children's thinking and their genetic development. For him, the logical sense of the world that surrounds the child arises only up to the stage of formal operations, that is, at the age of 12–13 years. According to his position, the development of a child's logical thinking is a spontaneous process, of natural maturation that only requires waiting for this moment to arrive.

Faced with this situation, the question that arises is how can mathematics be taught in primary school if the child does not yet have the logical component? If there is no logic, how is the child going to acquire mathematical knowledge?

The answer is that the only option of the child is to act automatically, to imitate, to repeat, and to memorize. Or, if the pupil of active school, the child just manipulates with toys, objects, and textures without having any idea of what he or she is doing. These lessons are called math lesson. This is what primary school does, covering the impossibility to propose new innovative method for teaching and development by the dogma of constructivist approach.

Fortunately, the constructivism is not unique conception of development of knowledge, base of the use of external and internal symbolic means.

3.4 Vygotskyan Perspective of the Formation of the Symbolic Function

The development of symbolic function is an essential line of cultural development of the child. This development in not spontaneous, but depends on the content of the actions, which the child fulfills in different period of ontogenetic development. The content of the child's actions depends on external practical, emotional, and verbal orientation provided by the adults, who take part in the child's actions. The development of the symbolic function requires the formation of mental representations of the characteristics of objects and problem situations. At first, the child is required to carry out practical actions with the objects. Then the child performs mental actions. Both practical actions and intellectual (mental) actions are determined by the correlation between the objects on which they are performed, between their properties and relationships (Rubinstein, 1964).

The symbolic function has the following characteristics: (1) identification of two plans (what is determined and what determines), (2) the ability to analyze the plan of signs, and (3) the ability to operate with the means of signs and symbols (Salmina, 2019). Symbolic functions do not exist as an abstraction but should be used in specific symbolic actions. These actions might be fulfilled on different plans of development. Firstly, symbolic actions might be fulfilled as external actions with objects (materialized substitutions), images (perceptive substitutions), and words (verbal substitutions). Later, symbolic actions can be interiorized and can take place as internal actions with internal images, internal words, and concepts.

Figure 3.1 shows as example of symbolic elaboration of the map of the "theater" with representation of rows and seats, marked with colors and number. The plan was elaborated by the pupil of the second grad of primary school, of the age of 7 years old. During the play, there are different actions. Some children buy tickets and sit in the corresponding seats. They identify their place by considering the number of rows and seats. Then they watch and enjoy the play.

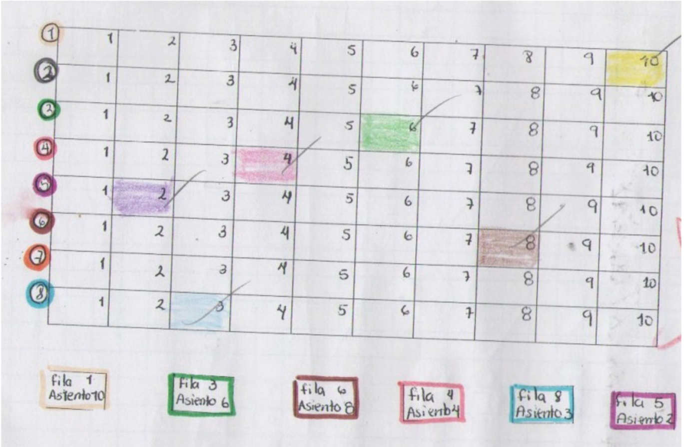

Fig. 3.1 Symbolic map of the "theater" (example from the notebook of the pupil Kepler School in the city of Puebla, Mexico)

Formation of symbolic actions might be considered as the part of general line of formation of intellectual actions by stages (Galperin, 2000; Talizina et al., 2017; Talizina, 2019). According to this line, it is necessary to start working with the child on the material level (real objects) or materialized (substitutions of concrete objects by symbols). The next step is directed to work on the perceptual plan (with drawing cards, images, and photographs). Only later, and through the constant support of the language (active pronunciation of what the child does together with the adult), the child acquires the possibility to act in the plan of oral or written language. Finally, the child will be able to act on the internal mental plan (conceptual or internal language). These stages refer to the steps of acquisition of intellectual actions with the use of conceptual knowledge (Solovieva & Quintanar, 2020, 2022).

Table 3.1 shows the line of intellectual development using intellectual logical actions of seriation in different plans, which might be used during the process of teaching and learning at the end of preschool age and at the beginning of primary school (Salmina & Filimonova, 2016). The teacher must identify the psychological development needs of children. In this way, the teacher can choose the plan of actions with the objects. For example, the teacher identifies that children require the use of real or materialized objects to carry out seriation. So the teacher uses toys of different sizes and begins the series: small, medium, and large. But if the teacher identifies that the children require moving to a more complex level of the series, the teacher can use materialized action with external symbols, for example, circle, square, and triangle.

At the same time, it is possible to offer the line of gradual formation of symbolic function in infancy. According to the line of symbolic development, it is necessary to start working with the child on the level of materialized substitution of one object by another (real objects and external substitutes). The next step is directed to work

Table 3.1 Plans of intellectual development

Plans of actions with the objects	Examples of intellectual actions
Material action with concrete objects	Seriation, based on the size of the toys
Materialized action with external symbols	Seriation, based on the shape of external geometrical figures
Perceptual action with concrete images	Seriation, by drawing the flowers, based on the colors
Perceptual action with symbolic images	Seriation, by drawing geometrical figures, based on the quantity of angles
Verbal oral or written action with no kind of external object or image	Oral or written seriation of any kind of objects, based on a chosen principle for Seriation

Source Self made

on the perceptual plan (substitution with drawing cards, images, and symbols). Only later, and through the constant support of the language (active pronunciation of what the child does together with the adult), the child acquires the possibility to act on verbal plan and achieve verbal substitutions by using words, phrases, and metaphoric expressions (symbolic verbal level). It is possible to suppose that during the period of primary and secondary school, adequate method of teaching might allow the pupil to arrive to the level of symbolic representation on the level of inner speech and inner images. Adolescence, probably, is the critical period of formation of this complex symbolic function. According to Vygotsky and his followers, this is the age of true interest for the arts, possible start of one's own creativity and interaction with entire cultural society (Elkonin, 1989; Vygotsky, 1995).

These stages refer to the steps of acquisition of symbolic actions with the use of different means actively and reflexively by the child.

Table 3.2 shows the line of symbolic development of the child with the examples of actions of symbolic substitution on different plans of realization of an action. These examples may help to imagine the sequence of development of symbolic function during ontogenetic and to propose the tasks, which require the action of substitution during the process of development, teaching, or correction.

Table 3.2 Plans of symbolic development

Plans of actions with symbols	Examples of symbolic actions
Materialized symbolic substitution	Table games or symbolic play
Perceptual symbolic substitution	Representation of the direction of movement on maps
Verbal oral symbolic substitution	Oral description of maps, schemes, and diagrams
Written symbolic substitution	Use of formulas, equations, structure of number, and so on
Mental symbolic function	Solution of mathematical equations, elaboration of tables and schemes

Source Self made

The symbolic function prepares intellectual actions as seriation, correspondence, and comparison, which are introduced explicitly and reflexively in the first grade of primary school (Salmina, 2017). Without the symbolic function, such actions are performed mechanically, repetitively, and without understanding its mathematical meaning.

According to research carried out by Salmina (2019), it has been shown that it is not necessary to wait until adolescent age for the child to acquire sufficient logical elements for learning mathematics. This is how it has been shown that the preschool-age child is capable of understanding, for example, the conservation of quantity, volume, and area (Salmina, 2017; Sidneva et al., 2021). At the same time, psychological research has shown that the period of preschool development should be dedicated to profound and creative playing activity, which allows to guarantee imagination, cognitive motivation, and broad social verbal communication of the children (Solovieva & Quintanar, 2019).

3.5 Learning Mathematics from Piaget's Perspective

From Piaget's perspective, children learn mathematics from concrete numerical concepts or the representation of quantities rather than general concepts (Piaget & Inhelder, 1971). According to the analysis provided by Kamii (2003) and Salmina (2017), Piaget established that the concept of number is a synthesis of the operations of conservation, classification, and seriation. These operations allow the child to identify the order and hierarchical inclusion between objects. For the child to assimilate the concept of number, it is necessary to differentiate the type of logical mathematical knowledge from the social one, because adults consider that the concept of number can be transmitted as social knowledge, which is characterized by being conventional. The main form of transmission is from the count that is carried out in social practice. However, Piaget (quoted by Kamii, 2003) establishes that the number is a mental structure that the child builds on his own and requires a lot of time for such construction. Also, it is established that the number is not a property of a set that remains invariable in the external world, but the number is an idea, that is, a child's thought. In this way, when the child has built the idea of a number, for example, "eight", he can produce a variety of symbols (e.g., 00000000, IIIIIIII) or with signs such as the spoken word "eight" or the spelling "8" and can even produce images. A symbol is a signifier that has a figurative resemblance to the represented object and can be invented by the child. In this way, the symbols do not need to be taught. Signs, unlike symbols, are conventional signifiers, they bear no resemblance to the object represented, and they are part of systems designed to communicate messages to other people. The word "eight" and the spelling "8" are signs that require social transmission.

According to constructivism, the social transmission or the teaching of mathematics consider in a simplified way the construction process that the child must carry out (Kamii, 2003). It is frequent to observe in the books dedicated to mathematics, which are used in the classrooms, the predominance of exercises of sets of images, and their relationship with quantities. Kamii (2003) points out that such exercise is based on the idea that children progress from the concrete level of objects to the semi-concrete level of images and from there to the abstract level of figures. However, directing students in that sequence is a major mistake, is an empirical conception, and reflects a lack of understanding of the difference between abstraction and representation. For Kamii, representation refers to the image. For example, the number three is represented by the numeral 3, or it can be represented by three apples in a drawing. Abstraction is a mental scheme as a product of mathematical logical operations.

So far, it is possible to conclude two contributions of Piaget's theory: first, the concept of number is an abstraction and not a representation, and second, the construction of said concept does not require teaching, and it requires the mental structures that the child is achieved from relating objects (logical actions).

Piaget, Kamii et al. (2010) have worked with primary school students under the premise that students are capable of inventing their own arithmetic without teaching. The authors raise the importance of encouraging students to think and generate their own ideas about how to add, subtract, multiply, or divide. In this sense, students always start with large numbers (tens) and then work with smaller numbers (units). Therefore, they conclude that the use of algorithms in teaching hinders students from developing their own ideas, and they fail to learn place value. This limits the possibilities of development of numerical meaning.

Following this line, the research by Kato and Kamii (2001) shows another example of how to use Piaget's theory in a Japanese school classroom. Children are asked to solve some mathematical problems without any kind of explanations or instructions, while the teacher observes how the child reasons individually and responds without receiving precise instructions. Subsequently, the teacher engages in a discussion with all the students, each defending and exposing their procedures. Considering the investigations of Kato and Kamii (2001) and Kamii (2003), the following premises are established for the teaching of mathematics from Piaget's theory: (1) the teaching of subjects must consider the three types of knowledge (physical, social, and logical), (2) the objective of teaching is autonomy, and (3) the teacher must modify his methodology as the student progresses.

3.6 Learning Mathematics from Vygotsky's Perspective

According to Vygotsky (1995), the process of teaching leads to development, but not every teaching may lead to development. Development occurs only if the process of teaching is carried out in the zone of proximal development (learning) of the child (Chaiklin, 2003). That means that the knowledge must be always new and attractive for the pupil, and that the child must be able to learn in collaboration with an adult. The word "orientation" is not used in Vygotsky's text, but it becomes the key point of the problem of teaching and learning in the works of Galperin (2000) and Talizina (2019). Orientation is understood is the central structural component of intellectual action, with the possibility of its external formation following by step-by-step interiorization. For its internalization, the use of material, materialized, perceptive and verbal means is required. These are external means that are gradually internalized with the help of the guidance provided by the teacher. The subject may interiorize an intellectual action and also may interiorize orientation base of this action (Solovieva, 2022). According to activity theory applied to teaching and learning process to learn means to fulfill different intellectual general and specific action, with the content specific to the area of knowledge (Solovieva & Quintanar, 2022). So, in the case of teaching and learning of mathematics, the pupils have to fulfill general logical action and specific logical actions that is mathematical actions.

This means that guided teaching makes it possible to learn mathematical skills (Salmina, 2019). For teaching of mathematics, the previous components are required, such as symbolic, logical, and numeric (Salmina & Filimonova, 2016). Some publications have stressed the importance of the inclusion of the component of spatial orientation, starting from the preschool age (Solovieva et al., 2010; Zarraga et al., 2017). The inclusion of the concept of orientation allows to work with the introduction of specific intellectual action, with symbolic general and numeric components in preschool age, when the leading activity if thematic social play.

Symbolic component is particularly important at the initial stage of preparation for learning of mathematics as special area of scientific knowledge. Symbolic component refers to the possibility of substituting one object for another, the means of representation (gesture, object, and action), the use of signs in actions, the use of sign systems (signs hierarchically related to each other), the symbolization of rules of behavior, the coding, the decoding, and the elaboration of schemes. In case of mathematics, this general level of representation is not enough. Symbolic component has to achieve the level of codification and de-codification, for example, as organization of one's behavior in role and rule games. Each game has internal hierarchical structure of rules, which might be represented by specific means (Lotman, 2001). De-codification of such means allows to accept and follow the rules of the play. Schematization and modeling are more complex levels of the use of symbolic function in intellectual actions and those can be used starting from primary school, specifically for each school matter (Davidov, 2000). Preparation for the use of schematization and modeling can take place at the end of preschool age, when the adults provide necessary conditions for organization and evaluation of children's intellectual activity (Veraksa et al., 2022).

The logical component as necessary element for preparation for learning of mathematics refers to the identification of various characteristics in objects, to the identification of general differences and similarities, to the identification of differences and similarities according to a given characteristic, to the need for comparison only according to an established parameter, and to conservation tasks with diverse content (bodies, liquids, distribution of space, area, etc.).

The spatial component is much close to logic component but has to do with the orientation on the level of relations between the objects on material, perceptive, and verbal level. Orientation in the real material space is previous in comparison with the orientation in perceptual plan (level of images) which should be introduced while working with the process of drawing in preschool age (Solovieva & Gonzalez-Moreno, 2017; Solovieva & Quintanar, 2019). The necessity of the work with graphic activity at preschool age might be argued as the necessity of preparation of spatial orientation on preceptive level and as the use of perceptive level of symbolization.

Verbal level of reflection of spatial relationships is fundamental for initial mathematics abilities. This level is complex and depends on reflective use of grammatical language structures (Luria, 1975). This level allows the child to point out the relations between the objects and processes and imagine their consequences.

At the beginning, the verbal level includes reflection of correlation of relationships between external objects, images, and body parts. The relations such as: higher than, less wide than, taller, bigger, the biggest, on the left side, between, among, the most, the less, and so on. Further, while understanding, speaking, and playing may achieve the use of relative, temporal, and cause/effect prepositions in the child's language. The tasks of feature identification as verbalization and showing various characteristics that objects have, feature differentiation and matching as verbal determination and identification of similarities and differences in objects and phenomena, object comparison as verbal determination of the basis, according to which objects are compared, allow to pass to the level of intellectual actions. This means that the use of objects and the identification of the characteristics of these objects allows the development of thinking skills such as comparing, contrasting, classifying, communicating learning because mental representations are generated. The object seriation and verbal determination of the whole process of the actions of seriation with basis, according to which the seriation is made and the possibility for variation of these parameters is one of examples of such intellectual actions. The grouping of objects with verbal determination of the basis, according to which the grouping is carried out and variations of these parameters is another example. The understanding of the conservation of essential physical features of the objects with verbal determination of the basis of conservation and of the characteristic that varies and the one that is conserved cannot be achieved with all previous mentioned actions. Let us remember that Piaget was specifically interested in conservation, but he never understood this process as specifically and culturally formed intellectual action. For him, it was a pure feature of high level of logical thinking.

During organization of the process of teaching and learning at primary school, which is directed to student's acquisition of mathematical concepts and actions, the teacher is required to have knowledge of the content, structure, and necessary characteristics of the actions that he or she intends to develop. The teacher must know how to organize the academic content of intellectual actions in stages: material, perceptive, external language, and internal language (Galperin, 2016a, b).

For example, during the process of formation of the concept of number, the teacher must organize the content and the actions of the pupils, which are necessary for complete understanding of the structure of number. Formation of the concept of number must be introduced starting from the action of measurement. The action of measurement contains the characteristics of the concept of number: magnitude, measure, and number of times of the use of the measure. The process of development of the action of measurement starts with the measurements of lengths, heights, width, distance, and magnitudes in different objects and conditions. In all cases, a feature of an object must be chosen as a magnitude to be measured together with election of specific external mean for measuring. Finally, the means for registration of the quantity of times for the use of the measure are also elected by children. The authors have used diverse situations of distances, which were drawn on the floor by different means. A piece of ribbon was chosen as a measure, and the quantity of measuring was recorded by drawing crosses. Finally, the pupil obtained the number of 5 times of the use of the ribbon for measuring of the whole distance on the floor (Rosas et al., 2017).

All intellectual actions were formed in the materialized level of action (as expressed in the previous example), later they were carried out with the schematic drawing (on perceptive level of action), and later with the writing of numbers and letters that represented (M = magnitude, m = measure and v = quantity of times). The last level was the level of external verbal actions, which allowed to pass to the use of algorithms.

This pedagogical proposal included the use of symbolic function. Latin letters were the means for identification of quantitative data in the tasks, as well as for identification of the mathematical relationship between the data (use of mathematical signs: greater than >, less than <, plus + , less –, by × and between /). At the end of the work with this kind of orientation, the students were able both to solve problems that were proposed to them and to create their own problems considering the data that was given to them (Rosas et al., 2017).

This means that the child's acquisition of new mathematical knowledge is carried out in the process of realization of actions with different objects in the constant company and orientation of the adult. The action is that unit that is used for the analysis of any learning process (Talizina, 2017, 2019). Galperin (2000) considers the action as a transformation of the object directed to the objective and based on an orientation. The child assimilates this knowledge, contributing to this process the content of his own intellectual experience and by interacting with the newly assimilated knowledge. The emergence of new knowledge makes it possible to provide a material that constitutes the basis of the child's mental activity (Poddiakkov, 1986). This newly formed knowledges are continually restructured and lead to the obtaining of next new knowledge because of highly complex forms of interaction with the newly established mathematical formations.

In addition, extensive internal cognitive motives may be formed, which represent an essential structural component of any type of cultural activity. This means that learning of mathematics at school constitutes a joint activity of collective interaction between the teacher and the collective of pupils. Internal motives are related to cognitive needs, when students are really interested and motivated by the content of learning mathematical concepts (Talizina, 2019).

The first mathematical concepts that must be introduced at school age are the concept of number and number system (Salmina, 2017). These concepts cannot be formed without the previous development of symbolic function on different levels:

materialized actions, perceptive actions, and verbal actions (Galperin, 2000). On the one hand, symbolic functions should be used in specific intellectual actions, which require the use of numerical material such as mathematical signs. On the other hand, symbolic functions should be used in general intellectual actions, which refer to logical and symbolic actions with great range of variety and flexibility. Both uses of symbolic representation are essential for mathematics. If the concept of number is not adequately assimilated, children will have serious difficulties for the subsequent study of the number system and, among other problems, for the understanding of the concept of number system itself (Talizina, 2019). For students to assimilate mathematical concepts and actions, the teacher is required to have knowledge of the content, the structure, and all the necessary characteristics of the actions that are intended to be developed in students. The teacher should be aware of organizing of the academic content in stages: material, materialized, perceptive, and verbal (Talizina, 2019).

Talizina (2019) considers that the concept of number is the first concept that the students face to understand mathematics as science. So, it is necessary to know what this concept consists of and even better continue to develop the understanding of this concept. According to Talizina (2017, p. 91), the concept of number is: "the relationship between what is subjected to a quantitative evaluation (such as length, weight, and volume) and the pattern that is used to provide this evaluation". Basic mathematical skills, as specific logic and symbolic general actions, precede the systematic acquisition of the concept of number and number systems. For example, the assimilation of the skills for the creation of signs and symbols, encoding and decoding of objects, identification of features of objects and their encoding in symbols, description of objects according to set of features with their representation in symbols, comparison of objects according to their features, and identification of essential features from nonessential ones (Talizina, 2019).

All these complex intellectual actions within mathematical tasks require the use of different signs and symbols (figures, letters, and schemes). For this reason, the voluntary and independent creation of symbols by children is necessary and obligatory as the main basis for introduction of mathematical knowledge at school. No kind of spontaneous process and no kind of natural experience may substitute the process of guided and orientated activity the signs and symbols must be included in the concrete activity for the solution of problems close to daily life and, later, for the solution of mathematical problems.

Figure 3.2 shows an example of the tasks, fulfilled by a child from the second grade of primary school. The task consists of measuring of the magnitude of length of different real objects in the classroom, such as the pillow, the table, the chair, the guitar, and so on. The child has to choose the magnitude to measure, the mean for measuring (e.g., the pencil, the stick, and so on), and to draw the schematic representation of number of measures, chosen for each object.

The concept of number depends on the magnitude that is measured (property of the object), as of the standard that is chosen to measure (measure). In addition, it is only possible to compare what has been measured with the same pattern. When asked what is greater, 5 or 7? The boy reflects on which measure has been used

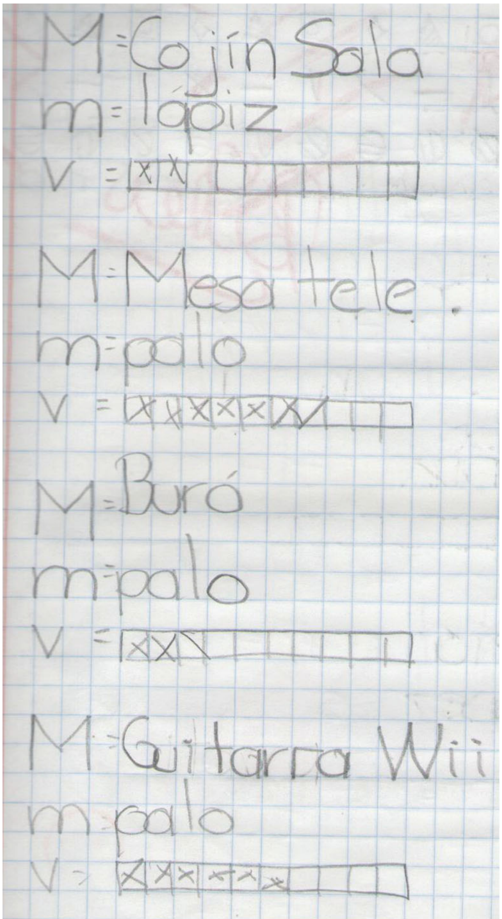

Fig. 3.2 Measuring task (example from the notebook of the pupil Kepler School in the city of Puebla, Mexico)

M: living room cushion, m: pencil.

M: TV table, m: stick.

M: donkey, m: stick.

M: wii guitar, m: stick.

because if you only consider the number in absolute form (number line) the student may have difficulties with more complex content (Talizina, 2017). It is necessary to identify the measure, in the case of question, what are greater, 5 or 7? The teacher can emphasize that they are 5 m and 7 cm, and then the student who has understood the measurements of length will be able to answer that 5 m because it has 500 cm while in the other set there are only 7 cm. This position principle of the numeral system allows the student to be able to solve arithmetic operations and explain your results. As we know, students cannot explain the conversion of measures; they only refer “he lends her so much” “he comes over here” “that’s what the teacher said”. The concept of number in relative form allows the understanding of the number system and other mathematical relationships, such as division, multiplication, addition, and subtraction (Rosas et al., 2017).

On the one hand, the teaching of mathematics from the logic of measurement enables the teacher to form the systems of concepts and to show the mathematical actions that arise from the different relationships between the elements of the measurement (magnitude, measure, and number of times required for the use of the measure).

On the other hand, initial learning of mathematical operations such as addition or subtraction should require external support from the objects and external manipulations. Later, the children can start to perform the same actions of addition and subtraction in the mind automatically (but with complete understanding of the sense of the action). According to Galperin (2000), the first plan of intellectual action is material external action. Finally, the idea of the action can appear which means that the actions have passed to ideal internal level.

Regarding the topic of solution of mathematical problems in primary school, the students must start by reflective determining of the actions he or she is going to carry out. To do this, reflexive identification of the final question of the problem and the conditions under which the question is established is required (Rosas et al., 2019). Symbolic function is obviously necessary for such reflexive identification. The role of orientation of the teacher is also essential (Burmenskaya, 2022).

3.7 Discussion

In recent studies, the teaching of problem solving in primary education was analyzed (Rosas et al., 2017, 2019). The authors have identified the elements of the content of the solution of the problems, such as: situation (what does the problem talk about?) and a question of the problem (what is asked in the problem?). Such elements might be provided to the students by the help of the orientation card. The card is firstly used by the students together with the teacher, after which the students carry out the actions independently. The actions, necessary for problems solution, were: (1) identification of the final question of the problem, (2) analysis of the situation and its relationship with the question of the problem, (3) identification of the necessary and unnecessary data to answer the question of the problem, (4) choice of mathematical actions and their verification, and finally (5) the answer to the question of the problem. Mathematical actions can be executed at different levels, according to the level of development of the students; that is, students can use counting sticks (materialized actions), draw a scheme (representative perceptive actions), write the algorithms (external verbal actions), or perform mentally (internal verbal actions). The authors observed that the students managed to solve simple and complex problems reflectively, in addition to the fact that the students create their own problems, anticipating the type of problem they will elaborate (simple, complex, and without solution). Throughout this solution process, the students intentionally use mathematical signs; for example, they decide with which letters to represent quantitative data. The whole research was conducted on the basis of the concept of orientation and guided intellectual activity.

All above mentioned shows that in cultural historical creative continuation of Vygotsky's perspective the symbolic value of the digit must be presented by the teacher (Salmina, 2017; Talizina, 2017); all symbolic and logical mathematical relationships of quantities are introduced for students as the content of oriented activity (Rosas, 2019).

According to all above exposed the revision of the perspectives of Jean Piaget and Lev S. Vygotsky is useful. Both authors agree about the importance of cognitive development and of special role of symbolic function.

According to Piaget's conception of intellectual development (Piaget, 1975), the child's own experience is fundamental to build its own knowledge of this experience. Cognitive development will depend on the situations in which the child participates directly, taking part in specific forms of cultural activity, organized by the adults.

According to Vygotsky (1992) and his followers (Talizina, 2019), cognitive development requires oriented and directed specialized teaching the whole content of programs of education in primary school must be revised (Solovieva & Quintanar, 2022). The teacher requires specific orientation in order to understand how it is necessary to orientate the children and how to as well as that the teacher knows the mathematical content of the work in the classroom.

The authors of the article believe that both perspectives should be studied reflexively to achieve important changes in the intellectual development of students. Some authors mind that Piaget's theory is the most widespread in research but unfortunately, not always it is properly used in day-to-day practice (Kamii, 2003). Vygotsky's perspective is difficult to assimilate and to carry out in educational organization, because it requires the change of the teacher before changing of the pupils. Both the teacher and the student are important actors in the teaching–learning process of mathematics and participate in an active, creative, and transforming way. Mathematics learning is not spontaneous. Children do not have their own resources to develop math skills. For the learning of mathematical skills, explicit and systematic teaching is needed. In this way, children learn, for example, the concept of number and establish the correspondence with the symbolic representation.

3.8 Conclusions

Mathematics abilities are not the result of some kind of spontaneous process. No child may construct independently mathematical abilities with no kind of symbolic, logical, and spatial actions previously formed. The teaching and learning of mathematics are a long process, which required specialized orientation, both for teacher and for the pupils. Both teaching and learning of mathematics is a kind of oriented cultural activities, which means that they must be formed during the process of assimilation of own cultural experience.

Teaching and learning of mathematics require the organization of actions that are related to general symbolic function; this function might be introduced starting from preschool age by the teacher. Without this orientation, it is difficult to learn the logical component and the mathematical knowledge that the child must assimilate at school age.

The teaching–learning process of mathematics at school age also requires guidance from the teacher, directed to the achievement of generalization, independence of execution, and reflection of own actions by the pupil. The actions should be introduced at different levels of action (materialized, perceptual, and verbal). Additionally, awareness of mathematical concepts and the way of how to use them should be developed during orientation in the concrete activity of solving problems, in which mathematical knowledge is required (Talizina et al., 2017).

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Chapter 4
Psychological Means and Mathematical Concepts Acquisition in Preschool Age: Which Are Better to Use?

Aleksander Veraksa, Nikolay Veraksa, and Margarita Aslanova

Abstract The conducted research is devoted to evaluating the effectiveness of the formation of mathematical concepts of size (length, area, and volume) in preschool children aged 6–7 years with different levels of development of executive functions. As part of the experiment, the concept of magnitude was formed using traditional, traditional with play elements, and symbolic and modelling approaches. The approaches were based on the same actions of children (selection of objects by size through comparison and measurement with the help of measurements), the same methods of their implementation, only the situations in which new concepts were introduced and worked out, and the means offered to children differed. It was found that the development of mathematical abilities in the course of formative classes occurs in students regardless of the approach used. This means that it is the actions of children organized by the teacher and the methods of these actions that are of fundamental importance in the formation of the concept of magnitude, and the situations and means of formation used play some other role, the meaning of which will be clarified in subsequent studies. The results of the study outline ways to further investigate the effectiveness of using traditional and non-traditional approaches in the formation of mathematical concepts in preschoolers. Results of the study are discussed in Vygotsky–Piaget perspectives on education and educational means in preschool age.

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4.1 The Types of Cultural Means Used by Preschoolers

Preschool educational programs often include some basic mathematics. It is partially related to the fact that children form their mathematical representations themselves, as was demonstrated by the works of Jean Piaget and his followers (Fleivell, 1967). However, it is important to emphasize the difficulties that occur while studying mathematics at school (Kesicioglu, 2021; Litkowski et al., 2020). It is not a coincidence that this complex discipline is present in the curriculum throughout all school years (Siegler et al., 2014). Social significance of mathematics for the development of thinking and mastering other disciplines justifies the adults' longing to facilitate understanding this science and to begin preparing children for its further mastering at school, in the preschool age (Stuart et al., 2013; Nguyen et al., 2016). Preschoolers learn to perform the tasks with comprehensible mathematical content. Therefore, in order to design adequate educational strategies for preschoolers, a study was undertaken aimed at the search for efficient approaches to the development of mathematical concepts in children.

However, concept acquisition cannot be analysed in isolation from consciousness development. While in the preschool age, it is representations, memory, and imagination that become pivotal, in the school period, thinking begins to play a more integral role. This transition is related by the mastering of the system of concepts. A concept takes a new place and obtains a new character in the general structure of the child's consciousness. Children's concepts develop along with other complex changes. The first group of changes includes the transformation of sign formations. The second group covers the changes in the system of sign formation within the structure of concepts. This is when arbitrariness, awareness, mediation, and systematic structure of concepts develop in children. Lev Vygotsky insisted that "the most significant in a child's development is that not only that separate functions of the child's consciousness grow and develop in the transition from one age to another, but it is also essential that the personality of the child grows and develops, the consciousness in general grows and develops. This growth and development of consciousness first of all manifests itself in the change of relationship of separate functions" (Vygotsky, 2004, p. 84).

There is a moment of objectivity in the logic of movement of the child's thinking even if he/she is still on the stage of precomplex thinking. Piaget believed that a child lives following his/her own internal logic that is gradually transformed by his/her social environment (Piaget, 1969). Vygotsky, on the other hand, indicated that from the very beginning, a child does not follow this internal logic. The reason for this statement is that a sign is initially included in an objective situation by adults. This is where all child's emotions and impressions get concentrated. The meaning of a sign partially coincides with the meaning an adult assigned to it which allows the child an adequate orientation in a real situation in order to understand and be understood by an adult. Moreover, Vygotsky emphasized that it is the adult that pre-sets the system of developing concepts where the child's consciousness arrives, sooner or later.

Thus, a certain generally accepted approach to the development of concepts in the childhood was established in the psychological science. This understanding is based on the word–image relationship. It is believed that the development of concepts in children follows the principle of generalization of different representations obtained within the process of interaction with adults in various situations. This approach to the definition of concepts can also be found in the works of contemporary authors. For example, "General ideas that organize objects, events, qualities, or relations on the basis of some similarity" (Siegler et al., 2014, p. 260). It is clear that this perspective is very close to the traditional understanding of this matter.

The studies organized by Venger and Kholmovskaya (1978) revealed that preschool age was sensitive to the use of visual models. A visual model is one of the forms of sign means. Its specifics as a psychological mean include the possibility of reproduction of such properties of reality in the model that allow children further orientation in the reality itself. Thus, with the help of the model, preschool children could find their way in the space of their group room in the kindergarten. They solved the problem of toy-finding checking the indications of its location on the model. The potential of visual models was explored through problem-solving related to the preschoolers' representations of different areas of knowledge. Schemes application also was shown too effective for mastering mathematics (Poland & van Oers, 2007).

DeLoach (2000) underlined the dual nature of visual models that represent particular objects and at the same time are related to other objects. From our point of view, this understanding of visual models is well-grounded. However, it is important to take into account that the characteristics of orientation within the model, and the orientation in the modelled real situation, often coincide. Therefore, a visual model is not that much of a symbolic mean (bearing emotion), but rather a sign mean. The opportunities of use of visual modelling in the process of intervention related to the development of mathematical concepts will also be discussed.

From our perspective, we distinguish signs and symbols. The main difference lies in the fact that a sign as a reference point of the object is initially aimed at its meaning when used, while a symbol rather implies the analysis of its external aspects and interpretation of its meaning. In other words, the orientation with a sign and the orientation in reality through using the symbol as a cultural mean are fundamentally different. We believe that the nonconcurrence of the external aspect of the symbol and its meaning is in fact, its most important characteristic. It requires an interpretation of symbol and complicates its direct use if compared to the sign. Besides, symbols are used in the situations of uncertainty and possess emotional load (Mamardashvili & Pyatigorsky, 1997; Veraksa, 2011).

Three types of tools were applied in our study for the development of educational strategies for the purpose of introducing mathematical concepts to the preschoolers. First, there were the traditional sign means reflecting image-related generalizations. Then, the visual models that also belong to the sign means. Nevertheless, it is the use of symbolic means that requires our special attention. As it was mentioned before, symbolic means have a dual objectiveness: orientation within the symbol and in the object represented by it are significantly different. The use of symbols in the preschool age can be often found in a play. Vygotsky (1966) mentioned the duality of preschool play. It manifests itself in the dual perspective, i.e. in the ability to see the substitute (the real object) and the substituted object (the imaginary one) simultaneously while playing. Thus, we can suggest that this duality of symbolization is quite comprehensible for preschool children.

4.2 Study 1. Mastering the Representations of Physical Phenomena

To compare the use of sign and symbolic means in preschool age, we turned to physical phenomena. The goal of our experimental study was the formation of representations of a rainbow in children. This formation took place in two ways: the sign-based, through explications and the use of models, and the symbolic, in the process of play activity.

The sample consisted of 4–5-year-old children ($N = 23$, 14 boys and 9 girls; $M = 56.5$ months) attending Moscow public preschool institutions from the district, designed for accommodation of middle class families. In accordance with the study goals, the participants were divided into two equivalent groups, namely the first experimental ($n = 13$) and the second experimental ($n = 10$).

4.2.1 Procedure

The Statue NEPSY-II (Korkman et al., 2007) subscale was used for the diagnostics of inhibitory control. Intellectual development was assessed by the Coloured Progressive Matrices (Raven et al., 2009). The participants were divided into two matched groups based on the assessment results with equal quantity of children who demonstrated high, average, and low level of cognitive abilities.

The key difference between them lied in the means for the understanding of the rainbow phenomenon introduced during the intervention. The first option was the symbolization of the key relationship of the phenomenon in the process of a play (the first experimental group), and the second, by means of schemes explaining this relationship (the second experimental group).

The intervention consisted of three stages. The content of the first and the last stages was identical for both groups, while the second stage was different.

On the first stage, the preschoolers were interviewed individually. They were shown a video with a rainbow. The experimenter asked the child to tell what he/she already knew about that phenomenon. Then, the adult played out a story with the help of two toys that focused on the rainbow. In the end, a rainbow, indeed, appeared, and the child was asked a question: "Why, do you think, the rainbow appeared?" In the end of the first stage, children were asked to draw this phenomenon and explain his/her drawing. All the answers were registered.

The results of the first stage demonstrated that the majority of children, while explaining the rainbow, only focused on the external characteristics, such as multiple colours or its shape, or expressed their emotional attitude to it ("Wow, it's beautiful!"). The analysis of children's drawings also revealed that all the participants put a lot of effort in the correct depiction of a rainbow: the number of colours and the shape. Normally, they did it with the help of a well-known mnemonic phrase about the hunter (Russian analogue of ROYGBIV) or relied on their previous experience (some children tried to remember what they saw in the video demonstrated in the beginning of the intervention). No one focused on drawing the components of the rainbow phenomenon such as sun, clouds, or rain. It means, children perceive the rainbow as something separate and not related to other natural phenomena. These results form an essential foundation and give the reason for the second stage of the experiment.

On the second stage, the content of work with the examinees depended on the group. Both groups were formed so that to match them by the results of previous psychological assessment as much as possible.

The preschoolers from the first experimental group were invited to play a special game where the key relationships of the rainbow phenomenon were revealed symbolically. The adult emphasized the fact that rain and sun were required for the rainbow to appear, and before the rain, it was supposed to be clouded. The game was organized in subgroups of three. Each participant was assigned a role (cloud, drop, ray of sun, sun, and rainbow) and was instructed to behave accordingly. Three plots were played out (the first one ended with the emergence of a rainbow, and two others did not). Each subgroup participated in two sessions with an average duration of 20 min.

The participants from the second experimental group were invited to individual sessions where they were explained the essential relationships of this phenomenon by means of a drawn scheme and modelling with construction blocks. The experimenter explained, demonstrated, and drew the same cases where the rainbow appeared or not.

The third stage took place 7 days after the second. The children were shown the same video again, and then, they were asked to draw a rainbow and to answer a number of follow-up questions about this phenomenon. All the answers were registered in detail.

Below, you can find the examples of the drawings (Fig. 4.1).

4.2.2 Results

The analysis of the drawings revealed that almost all post-experimental drawings depicted the conditions for a rainbow to appear (sun, rain, etc.). However, when asked about what exactly was happening in the picture, the participants were unable to answer.

Here are some typical explanations of the examinees:

The second experimental group

The first experimental group

Fig. 4.1 Examples of children's drawings on the third stage of the intervention

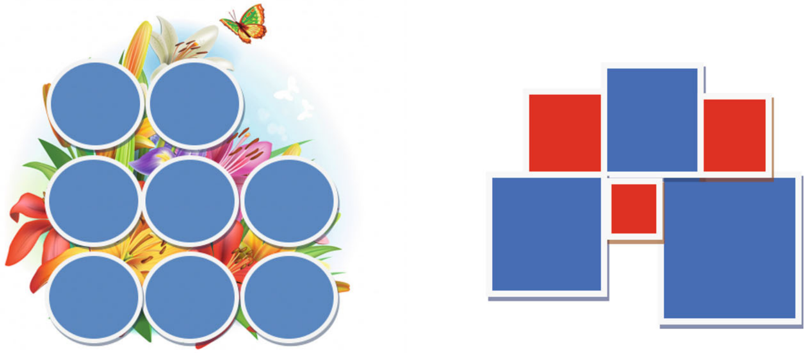

Fig. 4.2 Round and square "umbrellas" in the process of the exploration of the concept of an area

> Here, it is drizzling, then, the sun, the clouds and the grass, the sky… Why did I decide to draw them, too? To make it more beautiful.

> The rainbow is shining, the sun, and there are two clouds… It's just the background.

> This is an elephant, and something magical happened to it: he is watering from his trunk, and a rainbow came out… The sun was surprised… And the grass, and the clouds.

From our point of view, the children's comments indicated that the content manifested itself non-verbally, on the image level.

On the verbal level, children were basically unable to explain their drawings. It confirms that in preschool age, a child's activity is closely linked to the use of a symbolic image. It forms the foundation for this activity and for the depicting of a complex situation on a sheet of paper. In other words, children in their drawings reflected the structure that they intuitively (in Piaget's terms) captured by means of a symbol, but could not explain it due to the lack of conceptual (sign) means.

The design of this experiment was built on the strategy of activation of symbolization through creating a situation of uncertainty. The process of mastering of a phenomenon that 4–5-year-old children witnessed more than once became such a situation. But even despite being familiar with a rainbow, its phenomenon still remained a situation of uncertainty and cause orientation difficulties, since normally, children do not have a cultural behavioural pattern in the face of a rainbow, nor do they understand its physical processes. Thus, the experiment described in this chapter was aimed at the formation of representations of the rainbow phenomenon in preschoolers. This process followed two ways: the sign-based one, through the explanations and the modelling of the phenomenon, and the symbol-based one, in the process of play.

The results demonstrate the possibility of use of both sign-based and symbolic means for the mastering of a new learning content at the age of 4–5 years. However, what is truly important is that these means appear to be complementary: the former allow the reflection of the content under study, while the latter transmit this content in a fuller way, but not reflectively. The obtained results suggest the hypothesis that an efficient mastering of a new content requires a transition from symbolic to sign-based means in the process of learning. It allows the reflection and retaining of the content.

4.3 Study 2. Mastering the Mathematical Concept of Function by Means of Signs and Symbols

Our hypothesis was validated in the process of studying the effect of symbolic and sign-based means on the mastering of mathematical concepts by primary school children (the concept of function as an example). The level of intellectual development and means (sign-based and symbolic) were taken as independent variables, while the efficiency of mastering of the concept of a function became the dependent variable.

The sample consisted of the 3rd graders. In Russian general education schools, the concept of a function is studied in the 6th grade. However, this concept became the foundation for our experiment after a thorough research and consulting with math teachers. They indicated that for the 3rd graders, a function graph was a completely new knowledge far beyond their zone of actual development. Nevertheless, at this age, the children are already prepared enough to master it.

A three-week training program was designed. It consisted of 6 lessons, 40 min each. The program was aimed at mastering the concept of a function and its graphical representation.

Instead of focusing on the exact definition of a function, the main goal was to achieve the children's comprehension of it, i.e. the idea of a transformation of an independent variable (an argument) into a dependent variable (a value).

Two types of experimental classed were developed. The first was built on the use of symbolic image, and the second, on the traditional teaching methods.

In the beginning of the first lesson, both groups of students were given the same mathematical problem: "A bicyclist is moving from point A to point B at the speed of 10 km/h. The distance between the points is 50 km. If a bus moves at the speed of 40 km/h, how many hours it would wait in the point A before departing, so that it arrived to the point B together with the bicyclist?" Before the training, no student could complete this task.

We assumed that introducing a completely new type of a problem would put children in the situation of uncertainty when the use of already developed means is complicated. If corresponding images with a structure similar to the concept structure, this uncertainty can launch the symbolization mechanism. It would allow the transition from symbolic representation to the sign-based one.

4.3.1 Procedure

The first three lessons held in the first experimental group were aimed at the introduction of the symbolic image ("a magic wand") not linked directly to the mathematical concept under study but reflecting its structural interconnections. This introduction would promote the transfer of the main idea from its symbolic image to the mathematical concept itself. Our goal was to teach children to "discover" the idea of a transformation spontaneously and then transfer it to the concept of a mathematical function. To that end, an environment representing the transformation concept was required, so that the child could easily manage it.

Since transformation is the foundation of the very essence of magic, an image of a Magic Country was created for the first experimental group. There, two witches could do magic and turn things into anything. We believed that it would be easier for children to get emotionally involved in this story and that it would be fascinating to them to feel like magicians and be able to transform whatever they wanted, including themselves. With the teacher's help, sooner or later, the students realized that if anything could be turned into any other object, it becomes very difficult or even impossible to say what is which. The climax of the plot of our magical story was the first encounter of the two witches. This is when children were asked to help to solve the identification problem, since the witches could be anyone, and even turn into another.

We assumed that the biggest problem for children in this situation would be to explain the criteria of differentiation of witches. A play environment was supposed to facilitate the search for the solution and contribute to a more critical evaluation of the ideas that may emerge in the course of discussion. Children would be answering questions like, who were the witches? What could they do with their magic? How could they do it? One of the suggested ideas was that the wands could differ in their transformation capabilities. Children also gave examples of that difference: a transformation could be of different scale, a "strong" wand could do much more than the "weak" one. Thus, intuitively, children came up with the idea that the proportion coefficient was the power of the wand. The next step was the transfer of a symbolic image into the sign concept of a function. Then, children passed to the actual problem-solving. We should note that only a half of the lessons within the experimental program were dedicated to the solving of graphic problems.

The second experimental group that followed the traditional learning scheme was explained the concept of a function through a system of coordinates. Seats allocated in a movie theatre, chess, and similar arrangements were used as examples. Children got familiarized with the concept of a function through the introduction of values. Running and constant values were discussed (a person's height, the ceiling height, a distance covered, and a distance between two points). The intervals of variables and functions were explained as well. Same as in the first group, only half of the classes were focused on the problems where some vehicle moved that were solved by means of linear function graphs like $y = k \times x$.

The sample consisted of 49 children (23 boys, 26 girls; $M = 102$ months) from three different 3rd grade classes of a general education school.

Mastering of symbolic means including mathematical operation depends in major extent on the general intellectual development. This is why the diagnostics began with the Raven's Coloured Progressive Matrices (Raven et al., 2009). All the participants were divided into three categories in accordance with the obtained score. The sample was divided into two groups: the experimental ($n = 25$) and the control ($n = 24$) one. Each group included children belonging to all three categories in proportion matching their percentile frequency in the test results: high, medium, and low level.

After the last class, both groups performed a final test that consisted of five problems requiring a graphic solution. These problems were similar to the ones given when the mathematical skills of 6th graders are assessed.

4.3.2 Results

The obtained results revealed that children with high and medium level of cognitive skills differed in the efficiency of their mastering of the concept of function. The participants with high level of cognitive development generally solved four problems of six, while children with medium level solved three. The biggest difference was registered among children with low cognitive development. In the "symbolic" group, they were twice more successful than the students from the "traditional" group (3 problems against 2).

The goal of this study was to validate the following hypothesis: to explore the possibility of a transition to a sign-based representation of the concept of a function through operating with the content on a symbolic level. The intervention showed that the transition from the symbolic representation to the sign-based one is more efficient in children with a relatively low score in the Raven's Coloured Progressive Matrices (Raven et al., 2009).

This study confirmed the efficiency of the use of symbolic image for children who experience mathematical difficulties. It means that the use of symbols in the educational practice could be profitable for children who struggle to solve the problems by means of signs.

4.4 Study 3. The Use of Image-Based and Sign-Based Means by Preschoolers When Studying the Concept of an Area

We found ourselves facing the challenge of comparing the efficiency of symbols (as an adequate mean for the children with low level of cognitive development), visual model (as a mean that preschoolers are sensitive to, according to L. Venger), and the image-based generalizations commonly used in the learning process.

For the preschoolers in our study, we chose the concept of an area that is traditionally explained in the 2nd grade of Russian general education schools. This concept is not included in the preschool curriculum and therefore was a new content for all the examinees. The program consisted of 7 classes organized in mini-groups of three. This organization of the learning experiment allowed maximum involvement of all children in the process.

According to the conventional approach based on the use of visual image-based generalizations, children were offered different situations where the concept of an area was introduced through measurement units. The familiarization with the new concept began when the teacher suggested to view the area as an essential feature of any object occupying some space on a plane. Children learned to compare the areas of different objects superposing their images and explaining the differences in the space they occupied. Then, the teacher passed to some common measures represented by various geometric figures of different size. Thus, an object that took more space on the plane (the table surface) was measured by means of big squares, while a small object (a notepad) was measured with smaller figures. This approach helped children in the understanding that only those objects can be compared that were measured with the same units. Afterward, the examinees measured the same objects using different measures which allowed establishing the dependence of the results on the size of the unit. As an illustration to the problematic situation, the teacher explained that people agreed to use the same metric system to be able to understand and read the measurements. Finally, the concept of a square centimetre was introduced and presented as a nominal criterion. The square centimetres were also used to measure some objects, for example, small geometric figures.

In case of a symbolic image, we turned to the magical plot and witches again. They sent a heavy rain to wipe the small people's drawings from the asphalt. Children were asked to help the little people to protect the drawings with umbrellas (paper circles carved in advance). With the help of the teacher, children discovered that a circle could not protect the entire drawing, while square umbrellas were much more efficient. Therefore, the examinees were provided with paper squares of different size to use for protection. However, when they tried it, some parts of the drawings still remained uncovered. In the process of discussion, it was decided to make similar umbrellas and use the same squares. Children "measured" several drawings to find out how many little people with umbrellas they would have to call to save everything. When they got the difference, children, together with the teacher, drew the conclusion that the number of square umbrellas depended on the size of the drawing. On the next stage, children were asked to cover the drawing with small and big umbrellas to help them realize that the number of “small people” to come to the rescue depended on the size of the umbrella: the bigger it was, the fewer helpers were needed. Then, children measured the space around with the square umbrellas that they agreed to call square centimetres. Finally, they were shown a real square centimetre and were asked to relate its size to the size of the square in the notebook they used to measure the area of the figures.

The model was used to visualize the area and help children understand and master this concept. In the group that learned through visual modelling, children were introduced the concept of a scale in the first class of the program. They “built” a house on several square sheets to see that the size of the house depended on the size of the squares. Then, increasing and reducing the size of the “blocks” children changed the size of their building. On the next stage, children worked with a model of the room where the class was taking place. Small blocks represented the furniture, and below, the examinees could find an instruction indicating where to look for a star. Having found it on the maquette, children were to find it in the real room. Then, the concept of area was introduced as the number of squares occupied by the object. Children were offered different real objects (notebooks, pads, and erasers), so that their area could be measured. The silhouette of an object was drawn out with a pencil which allowed counting the squares that would cover it. Eventually, the concept of a square centimetre was presented. Afterward, children were asked to calculate the area of an object in square centimetres based on the number of squares the object occupied in the notebook. Before, the squares were measured with a ruler, so it was already known that it equalled 1 cm^2. However, at this moment children were also explained that one square could be both 1 cm^2, and 1 m^2, depending on the scale. Moreover, the examinees were asked to solve some engineering problems. For example, to build a hangar for a plane of a certain size or to decide if a boat and a car drawn on a gridded sheet of paper would fit into their garages drawn on a gridded paper with bigger squares.

4.4.1 Procedure

The sample consisted of 100 6–7-year-old children from Moscow kindergartens (*M* = 6.5 years, 43% boys, and 57% girls).

Cognitive skills were assessed with NEPSY-II (Korkman et al., 2007) scales for visual memory (Memory for Designs), auditive memory (Sentence Repetition), switching, inhibitory control, and cognitive flexibility (Naming and Inhibition; Dimensional Change Card Sort (Zelazo, 2006). Children with low, medium, and high cognitive level were evenly distributed among the experimental groups.

Before the experiment, all the examinees were tested individually to find out if they understood the concept of an area. The children were given 9 tasks. Each contained a picture with two lines of connected figures to be compared. These tasks were designed specially for this study. In 8 of 9 tasks, the figures were different in

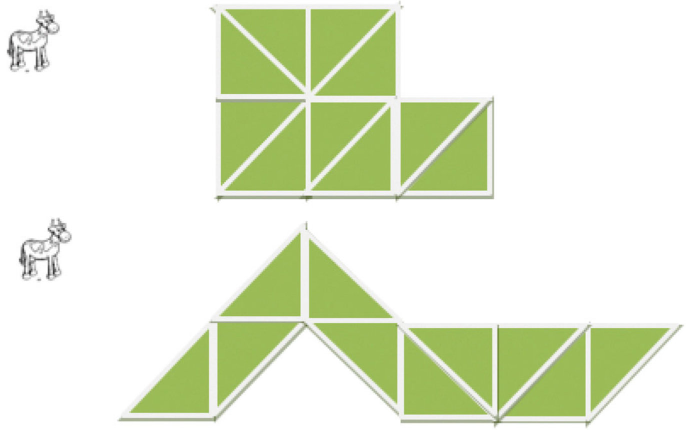

Fig. 4.3 Example of pre-test task on the evaluation of areas

shape, number and/or position, but their total area was the same. Children were asked to answer the question: "Do both cows have the same amount of grass to paste on, or does one have more?" All the responses and comments if there were any were registered in a special form. We assumed that a child with a well-developed concept of an area would be able to see that the grass area was equal in each picture (Fig. 4.3).

One week after the above-described classes, post-test was held. In the first part of it, children were given the same nine tasks they solved before the experiment. It allowed the assessment of the development of the children' representations of an area. Afterward, four more tasks were offered, aimed both at the assessment of the understanding of an area and the achievements in the zone of proximal development. To this end, children were to solve problems with 3D objects, where not only the area of the objects was to be taken into account, but their volume as well (Fig. 4.4).

Taking into account that before the test, the groups were comparable by their performance in the tasks, we could evaluate the efficiency of the training programs for children with different level of cognitive development.

4.4.2 Results

The experiment revealed that the children with high level of cognitive development were more successful than their peers with medium and low level when they studied the concept of an area through modelling (Table 4.1). Children with low level of cognitive development it was symbolic means that worked the best. Interestingly, that in the "symbolic" experimental group, these children in fact demonstrated the most noticeable progress.

Our results show that children with low cognitive abilities can in fact demonstrate the biggest increase in their knowledge while studying mathematical concepts in

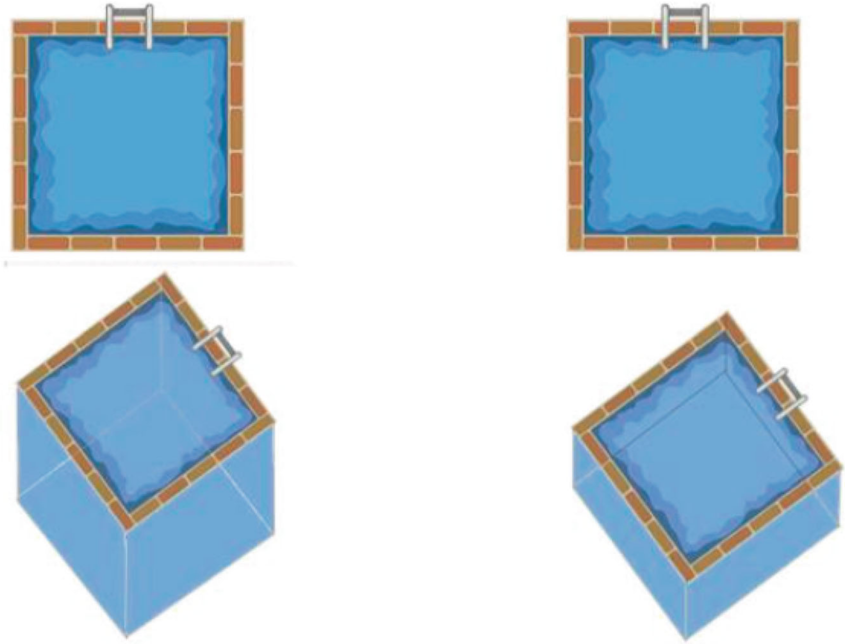

Fig. 4.4 Example of the additional post-test task for the assessment of the development of the concept of an area. The question is, which swimming-pool has a bigger capacity?

Table 4.1 Mean post-test values of children with different level of executive functions development

	Approach	Total post-test score $M \pm$ SD
High level of EF development	Traditional	7 ± 1.93
	Symbolic	7.4 ± 3.36
	Modelling	*7.46 ± 3.43*
Medium level of EF development	Traditional	*7.2 ± 2.66*
	Symbolic	6.08 ± 2.99
	Modelling	6 ± 3.81
Low level of EF development	Traditional	6 ± 3.64
	Symbolic	*6.11 ± 3.26*
	Modelling	4.2 ± 3.271

Note The maximum score for each level of EF development on every stage of final test is marked in italics

groups (e.g. the concept of an area) through symbolic means, i.e. when the concepts are introduced in an emotionally structured narrative. And vice versa, in a traditional framework, their mathematical skills develop much slower.

This outcome is relatable to the data obtained in the previous study and supports the idea of use of symbolic means in the mastering of mathematical concepts. This conclusion can be explained by a natural use of symbols in play which once again emphasizes its value for the educational process. The appearance of “play-based learning” confirms that the apologists of this concept intuitively understand the importance of the unity of cognitive and emotional aspects of play which allows it becoming an educational technology.

4.5 General Discussion and Conclusions

The results of the studies demonstrate the efficiency of symbolic means in the mastering of representations of physical phenomena and mathematical concepts, if compared with sign-based means. To a great extent, the possibility of use of symbolic means in preschool age is related to the principle of dual coding intrinsic for play activity. There, the child has to retain two realities, the imaginary situation and the visible plane (Lillard, 1993).

Here, it is important to emphasize the specifics of play activity. As Leontyev (2000) indicated, play is a modelling activity in the first place. The child tries to reproduce the situations that cannot be experienced in real life, or it is complicated (e.g. horse-riding). Acting with a stick, the child seeks to understand this exciting situation as if he/she was indeed riding a horse.

In other words, in respect to the use of symbolic means in the logic of dual coding, we should indicate that their emergence in play is possible because of the specifics of the very play situation. It is built on the need of the child to understand some real-life situations of the adult world. The uncertainty of the explored situation and the opportunity to fulfil that need, even if on an imaginary level, create the emotions that drive that permeate play activity. Later, the works of Diachenko (2011) demonstrated that play activity is of symbolic nature in the sense that it engenders an imaginary situation where real objects obtain new play-related meanings.

All this allows the application of the following technology when teaching mathematical concepts to children. First, a symbolic (imaginary) space should be constructed with its characters and their attributes that help to create situations with different symbolic contexts. It is important to find such symbolic context that would as close to the objective meaning of the real concept as possible. For example, in our study, in case of the mathematical concept of a function, the magic wands of two witches had different power to increase the objects. That power allowed children the transition from the fairy-tale context to the mathematical. Because of the fusion of symbolic and sign-based contexts, a magic wand could be transformed into a mathematical object. The children understood the concept of a function due to their dual vision of the situation (the unity of image-based and sign-based representation). Of course, it is clear that only the first step was made, since, as Vygotsky put it, a complete mastering would require the deployment of the entire conceptual system (Vygotsky, 1991).

In the study of the understanding of an area, it was an umbrella that was transformed into the concept. First, children regarded it as a simple mean of protection against the rain, but gradually, it began to encapsulate the whole idea of protection from water, not only for people or objects, but for some space. It allowed further transformation into the unit of area measurement.

Thus, we could see the logic of use of symbols. Initially, it appears in its symbolic meaning as an object of an imaginary (a symbolic) situation. Then, its function in this situation is defined. This function is discussed with children and gets transferred to the space of mathematical relationships. Afterward, this transformed symbolic function helps to construct a mathematical concept. It is important to emphasize again that in this situation, both symbolic and mathematical sign-based space are present and acknowledged.

Our studies demonstrated that different types of means can be used to explain mathematical concepts to children. Among them, there are traditional image-based generalizations, visual models, and symbolic means. To us, visual models and symbolic means are fundamentally different. In the first case, the character of the child's orientative activity both in the model, and in the way of its use in the modelled reality, is generally the same to a major extent. Of course, we can discuss the double objectiveness of a visual model, but it is very distinct from the dual objectiveness of a symbol. In the former, the recoding is performed consistently and progressively while the child moves from the model to the substituted reality. In case of the symbol, the child has to retain the dual perspective considering both the symbolic content and the content of the symbolized object. This experience of a simultaneous dual representation is obtained by children in play activity. Thus, they master a powerful tool of analysis of reality in the situation of uncertainty. It is the symbolic means that allow preschoolers distinguishing the symbolized means that are adequate for the reality on the one hand and simultaneously remain in the content of the symbol and the newly discovered meanings, on the other.

As our results show, symbolic means have a special resource that can be of support for children that struggle in the use of sign means when mastering a new content brought by the situation of uncertainty. They help to master complex mathematical concepts already at preschool age and thus promote better progress in the studies of mathematical content at school.

In regard to the mathematical skills development, this step consists of daily real-life concepts of a child allowing to exploring the symbolic meaning of the key object and its transformation into a mathematical one. Symbolic and mathematical spaces are united, and the moves within them are performed within the scope of dual perspective.

Another important specific feature of image-based and sign-based representations of learning content should be taken into consideration. As our studies revealed, when primary school children have to express the content of the disciplines they study through image-based representations, they can experience difficulties in the transmission of the meaning of their activity. Meanwhile, the distinctive feature of sign-based representation is that it allows the reflection of its content. An image-based representation, though, allows retaining the content of the discipline, but it is not transmitted reflectively.

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Chapter 5
The Pre-number Period of Teaching Mathematics in Russia: A Psychological Analysis from the CHAT Standpoint

Anastasia Sidneva and Valeria Plotnikova

Abstract The objective of this chapter is to discuss the principles developed in the mainstream of the Cultural-Historical Activity theory (CHAT) for analyzing the goals of the pre-number period of teaching mathematics in Russia from a psychological point of view. We have shown that the most important issues in mathematics education concern the type of concepts proposed for assimilation in this period; the type of students' actions by which these concepts can be assimilated and practiced; and kinds of means to carry out these actions which are provided. According to these principles, it was developed an original program for teaching mathematics in the pre-numerical period (15 experimental sessions, 15–20 min for each session, mini-groups of 3–4 children). We started introducing children to mathematical reality by having them master an elementary mathematical concept such as magnitude (length, area, and volume), which can help them understand the dependence of number on measurement. The control group did not attend any special sessions but learned some mathematical concepts and skills as part of regular school preparation classes conducted by their teachers. The total number of participants was 37 (18 in the experimental and 19 in the control group). Before and after the sessions, each participant was tested on their mastery of the mathematical concepts and skills. The results demonstrated that the children who attended the experimental program showed a significantly greater increase in post-test scores than the children from the control group. This occurred in their mastery of the concepts of length and area and the dependence of number on the means of measurement, and their ability to select values and assemble "complex" sets. The results have proved the effectiveness of the program built in accordance with the CHAT principles.

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5.1 Introduction

The educational standards of many different countries (Federal State Educational Standard for Primary Education, Russia, 2014; Performance Indicators in Primary School, Australia; The Common Core State Standards Initiative, U.S.A., etc.) include a block on mathematical development and the acquisition of elementary mathematical concepts by 5–7-year-old children. The concepts include orientation in space and time, shape, size, quantity, and magnitude. However, in most kindergarten and school math programs, the educational content focuses on numbers and their operations (addition, subtraction, etc.) and only systematizes the child's previous experience with magnitudes and shapes, without introducing anything new (Cheeseman et al., 2018; Veraksa et al., 2022).

This situation may be due to the widespread view on the importance of developing a sense of number in childhood for the further mathematical achievements. Several studies have demonstrated that nonverbal number sense is correlated with mathematical abilities in children and adults (Jordan et al., 2022; Starr et al., 2013). But it is essential to emphasize that the child's initially intuitive number sense appears when he or she interacts with real objects and, primarily, different magnitudes.

Most of the existing educational programs for starting to teach mathematics in Russian schools (7-year-old children, first grade of primary school) include a special period called a pre-number period. This unit precedes the study of the number sequence and operations with them and includes the generalization and repetition of mathematical content that the children mastered in preschool childhood. We have not found such a special period in the educational standards and programs of other countries, where the learning of mathematics begins with the parallel mastering of number skills (counting, number recognition, number naming, quantity comparison), measurement, and the basics of geometry (Heuvel-Panhuizen, 2000; MacDonald & Murphy, 2021; Solovieva et al., 2014, 2021a, b, 2022; Wang et al., 2017).

In addition, the existing instruments for assessing the mathematical competence of young children in other countries mainly assess numerical development, and to a lesser extent, include tasks testing the children's abilities in the areas of measurement, shape, patterns, etc. (REMA by Clements et al., 2008; Numerical Abilities Test by Beltrán-Navarro et al., 2018; Preschool Numeracy Indicators by Hojnoski et al., 2009; Preschool Early Numeracy Scales by Purpura et al., 2015; AEPS, Gao & Grisham-Brown, 2011). This fact emphasizes that common primary mathematical education focuses on the numerical period.

However, the effectiveness of the subsequent assimilation of basic mathematical concepts obviously depends on the pre-number period. The objective of this chapter is to analyze the content and goals of that period in teaching mathematics in Russia from the CHAT perspective, and to show how it is possible to design a program for teaching mathematics in this period. It is worth mentioning that CHAT is widespread not only in Russia, but also in Europe and USA (Freiman et al., 2021; Oudes-Slob et al., 2022; van Oers, 2012). Therefore, it can be assumed that foreign programs built on the logic of CHAT include a pre-number period of mathematical education.

5.1.1 The Content of the Pre-number Period of Teaching Mathematics in Russia

The main content of the pre-number period in different programs lies in acquainting the children with the characteristics of objects (color, shape, and size) and comparing them with spatial representations (top, bottom, left, and right); with the simplest temporal representations (earlier, later, first, and then), and with the counting of objects (using quantitative and ordinal numerals) (see, for example, Bantova, 2017; Arginskaya and Kormishina, 2016). Some programs during this pre-number period concentrate on the concept of magnitude (length, area, volume, and others) and the ability to arrange objects by magnitude (seriation) (Dorofeev et al., 2020); sets (with the introduction of the concept of "set" and "element of set") (Dorofeev et al., 2020; Peterson, 2019); addition and subtraction of sets (Peterson, 2019); and the ability to classify and group by attributes (Peterson, 2019). The duration of this period ranges from 5 to 120 h, depending on the program.

The largest number of hours in the pre-numerical period is allocated in programs built in accordance with the principles of CHAT, specifically, the theory of the Developmental Instruction (DI) designed by V.V. Davydov (Aleksandrova, 2011; Davydov, 2000; Davydov et al., 2011). Such a wide variation is due to the fact that different programs have different goals for teaching mathematics in the pre-number period. Most often, the main purpose of studying mathematics in the pre-number period is to systematize the children's preschool mathematical experience, including their experience in kindergarten, since approximately the same ideas are the goal of mathematical training in preschool educational institutions (Veraksa et al., 2019). But at the same time, this period should be the basis for further study of numbers and other lines of required mathematical education in elementary school (arithmetic operations, text tasks, geometric shapes, and geometric quantities, etc.).

5.1.2 The Goals of the Pre-number Period According to CHAT

From a psychological point of view, there is a fundamental difference between this period of studying mathematics at school and the preschool stage of mastering mathematical concepts. Vygotsky wrote about this fundamental difference in the content of the concepts offered in school education from preschool (Vygotsky, 2004). He noted that most researchers distinguish school education from that in preschool by its special organizational character: school education is organized, and preschool education occurs spontaneously (Vygotsky, 2004). School education and preschool education have different relationships to the process of mental development.

But, according to L.S. Vygotsky, the "*properly organized education of a child leads to a child's mental development, [and] brings to life a number of such development processes that would become impossible outside of learning at all,*" and develops not his natural inclinations, but the "*historical features of a person*" (Vygotsky, 2004, p. 345). Also, each academic subject has its own specific relationship to the course of children's development (*Ibid*., p. 347). In the field of mathematics, this issue was developed in detail by Davydov and his colleagues (Davydov, 2000). Davydov was fundamentally opposed to the idea that the learning process is simply the assimilation of knowledge; this author believed that the type of knowledge to be acquired is determinant and that it differs fin the school and preschool periods (Davydov, 2000).

According to the previously mentioned, important questions may arise.

What kind of developmental processes are brought to life by school education, specifically in mathematics? How should the propaedeutic stage of its assimilation be organized? The purpose of our further analysis is to discuss the functions of this stage from the CHAT point of view: first, the adaptation of the curriculum for the assimilation of mathematical concepts, and second, its impact on mental development.

A detailed analysis of preschool mathematics compared with school mathematics was given by Davydov in the book "Types of Generalization in Teaching." This author has shown that in most modern mathematics education programs, there is a complete "continuity" with the child's preschool experience, which is manifested in the fact that "primary education is a natural continuation of preschool, [and] actively uses and assimilates the knowledge of children acquired before school, in particular, knowledge about number and counting" (Davydov, 2000, p. 40).

According to Vasily Davydov, a preschooler's mathematical representations are not based on the relationships between objects that are essential for this subject area. Thus, the understanding of numbers only as a result of counting of material objects (usually concrete objects as isolated units), which is always used in preschool practice and often underlies the content of first-grade programs, is actually far from a scientific understanding of numbers, which allows us to act intelligently and meaningfully in the field of mathematics in the future. Such an understanding (called "theoretical" by Davydov, 2000) conceives a number as the result of measuring a quantity by an adequate measuring tool or measuring mean (such a value can be not only the number of objects, but also length, perimeter, area, volume, etc.).

By relying on continuity with their preschool experience with each mathematical concepts, we are only introducing children to a special case of a number. This naturally leads them to make errors (e.g., attempts to add/subtract different-quality quantities such as the mass and number of objects, misunderstanding the meaning of decimal and natural fractions, etc.).

Thus, the first goal of the pre-numerical period of teaching mathematics should be the children's assimilation of a concept essential for further meaningful study of numbers: the concept of magnitude and measurement (the aspect of the magnitude by which it can be measured). Among such values are length, area, volume, etc.

The second important aspect is the actions by which such concepts should be acquired and assimilated by the students. Lev Vygotsky pointed out directly that scientific concepts are not the result of generalizing about objects by observing their everyday characteristics; they are set "from above" (Vygotsky, 1934). Another important difference in the actions for the assimilation of scientific concepts has been explained in the work of Leontiev et al. (Gal'perin & Talyzina, 1957; Zinchenko, 1961; Leont'ev, 2003): the concept of being set "from above" will be assimilated qualitatively only if the child understands that it is necessary to use it in a certain way, that is, if it becomes the subject of his activity (Zinchenko, 1961).

In fact, the actions the teacher provides for the introduction of these concepts should convey to the child the need for a new concept: that is, allow him/her to see the task which led it to arise. According to Davydov, the actions used for the introduction of concepts should be practical but performed for educational purposes (in order to discover a generalized way of action) (Gal'perin & Georgiev, 1960; Gal'perin & Talyzina, 1957; Davydov, 2000). They should be based on the generalized method, not so much for the purpose of its "application" as for the purpose of its concretization or definition of boundaries (Vysotskaya & Rekhtman, 2012).

At the same time, such actions should "work" for the formation of (a) consciousness (the ability to explain what and why I am doing), for which naming (the speech form of action) and modeling are important; (b) generality (for which variations of task types are important); and (c) assimilation in the mind, involving gradual reduction and integration into other forms of activity.

The third important aspect is the type of means for carrying out these actions which are provided. Such means should be considered psychological cultural tools that allow one to master new types of activity (Vygotsky, 2004). Mastering the cultural system underlying one's own cognitive activity is an important aspect of cognitive development (Elkonin, 1989; Karabanova, 2005; Venger, 1986).

5.1.3 Description of the Experimental Program

Studies done from the CHAT perspective have shown (Aleksandrova, 2011; Davydov, 1966; Davydov et al., 2011; Obukhova, 1972) that it is advisable to start introducing children to mathematical reality by mastering an elementary mathematical concept such as magnitude. Magnitude itself is usually determined using three comparative relationships (i.e., $a = b$; $a > b$; $a < b$); examples of magnitudes that preschoolers constantly confront include length, area, volume, and quantity. The concept of magnitude is essentially a system-forming concept that underlies the concepts of number, function, and figure, and, accordingly, links three domains of mathematics: arithmetic, algebra, and geometry (Davydov, 1966). According to Galperin's theory, any concepts new to the child should be learned as reference points for relevant actions and reflect the conditions under which these concepts originated (Gal'perin, 1975).

Some studies have shown that from both the psychological and logical/subject-related standpoints, the most complete and appropriate idea of magnitude is formed when the child learns actions of comparison (establishing the correspondence or non-correspondence of magnitudes) and measures quantities using a conditional measuring tool (how many times the measuring instrument fits into the given magnitude) to establish the relationships between them (Davydov, 1966; Gal'perin &

Georgiev, 1960). The use of a conditional measuring tool makes it possible, first, to compare objects that cannot be directly placed upon each other, and second, to concretize the relationship between magnitudes and understand how much one magnitude is larger or smaller than another. When teaching is organized this way, the concept of "magnitude" is mastered as a necessary reference-point for a specific object-oriented action—the action of comparison and measurement—which allows us to determine whether the children have understood its essential features.

In our program, the concept of magnitude was introduced and worked out through the task of helping an imaginary character (e.g., "pour the same amount of the life-saving water to save the queen," "help Winnie the Pooh find his way home from the dark forest"). This is justified by the fact that within the CHAT framework, it is emphasized that the imagination that develops in the children' play (Elkonin, 1989; Vygotsky, 1966) plays a key role in the development. Recent research shows the importance of symbolic tools included in the play for mastering initial mathematical concepts in preschool age (Fleer, 2022; Li & Disney, 2023). Thus, in our program, we created emotional meaningfulness for the action. The means for solving the problem situation were also symbolic objects (e.g., a magic ball for measuring a route; umbrellas that can protect a drawing on asphalt from rain; a magic cup). These symbolic representations (Solovieva et al., 2013) established the problem situation, key points of orientation, and relationships for its solution (Rosas & Solovieva, 2018, 2019; Solovieva et al., 2016), becoming reference points for mastering the concept of magnitude. In this case, the children did not need a model of the action; they themselves could construct the necessary action based on the symbolic image of the situation, since the symbol as a cognitive tool facilitated perception of the conditions for performing the task in a situation of uncertainty (Veraksa et al., 2015). So, for example, you can set the following task for the children: "The gnomes drew beautiful drawings with crayons on the pavement. But it seems it is starting to rain. Let us try to protect these drawings from the rain. Which of these umbrellas can help us?" Children are offered square or round umbrellas. Umbrellas, which allow you to cover the entire surface of the drawings, in this case are a symbolic display of such a quantity as area.

The contents of the original program are presented in Table 5.1.

This program is intended for Grade 1 students as an introduction to mathematics. However, we found it necessary to evaluate its effectiveness on older preschoolers immediately before their admission to school.

Table 5.1 Contents of the pre-number period' program

No. of session	Contents	Example of the tasks	Solution
1	Length. Introduction of the indirect comparison	Evil Witch hid the magic wand case in her cave and multiplied it, so now it is unclear where the real wizard's case is. The wand should fit perfectly in the case. It should not be shorter or longer, otherwise it will not charge. Witch allowed to enter the cave once. If the first time you find the right case among its copies, you can pick it up. But you can enter the cave only without a magic wand How can we find the right case?	Children need to measure a thread of the same length as a magic wand and take it with you to the cave
2	Length. Indirect comparison by length. Training	Gnomes hold competitions for the longest jump. Let us determine who jumped the furthest	It is necessary for each dwarf to attach and cut a thread of the same length as the length of the jump
3	Length. Measurement using a conditional measure	There is a competition for the longest tunnel in the gnomes' village. The winner will get a delicious cake. Let us find the longest tunnel. *Note: the tunnels are too long and winding to be measured by one big measure, so it is necessary to use small conditional measure*	Children need to measure the length of each tunnel with a small measure
4	Length. Measurement using a conditional measure. Training	As a token of gratitude, the gnomes decided to give the Prince a throne. And that is how they measured the throne with small staffs. Look, which one of them did it right?	Children should understand that they need to choose the measurement of the same size and that there are no distances between the measurements when measuring

(continued)

Table 5.1 (continued)

No. of session	Contents	Example of the tasks	Solution
5	Length. Measurement by different measures. Standard length measures	The gnomes wanted to know whose house is higher. Here is how they measured the height of the houses (*different conditional measures are used in the figure*). But they could not find out the answer and they quarreled. Why? The dwarves argued and quarreled many times because each time they got different answers. Although they measured everything with staffs and measured correctly. But each gnome took his own staff. So, these quarrels and swearing really bothered the Prince. He wanted to come up with something, so that no one would quarrel over how and what to measure. What could the Prince come up with?	Children should understand the need to use the same standard-length measurement
6	Square. The tangram method for the square measurement	In a magical land, there are magical items. For example, clothes that make you invisible Our friend, Wizard, asks us to help him make a magic cloak. But there is an important condition, there should be as much fabric as in the King's mantle, and the new cloak should occupy the same amount of space as the King's mantle. Otherwise the magic will not work. How can we do this?	A tangram is a figure consisting of smaller figures, when moving which the place occupied by the tangram (its area) remains the same. The children must make a cloak that will take up as much space as the King's mantle
7	Square. The tangram method for the square measurement. Training	*Children are offered a drawing of a magic flying carpet* This magic carpet has a brother. They occupy the same amount of space. However, Evil Witch stole it. And now we need to find him *In the picture, there are several carpets of different shapes. One of them is the same in area as the reference carpet*	Using the tangram method (cutting and rearranging parts of the figure), children must find a magic carpet of the same area
8	Square. Measurement using a conditional measure	Funny little men drew pictures on the asphalt with crayons. But Evil Witch sent rain. How can we protect the pictures from the rain?	Children can use square-shaped umbrellas (units of area) to cover the entire drawing

(continued)

Table 5.1 (continued)

No. of session	Contents	Example of the tasks	Solution
9	Square. Measurement using a conditional measure. Serialization of objects by square	The little men drew a few more pictures and decided to arrange them from the smallest in area to the largest. Let us help them	Using area measurements, children need to measure the area of each figure and find the smallest one first, then the larger one, etc
10	Square. Measurement using a conditional measure. Standard square measures	It so happened that funny little men measured how much space the picture took up, while others built a tent of shields for it. But then the tent did not fit the picture because the men used different shields for measuring. (*Show this situation on paper and billboards*). Let us help the funny little men not to quarrel anymore. What can we come up with?	Children should understand that it is important to use the same units of area measurement to measure the space occupied by the picture and the space occupied by the tent
11	Volume. Indirect comparison by volume	To save the Queen, you need to pour as much living water into a crystal vessel as in a golden bowel. But it is impossible to carry a crystal vessel and a golden bowl across the river separating them. How to get the right amount of living water and save the Queen?	Children need to choose two identical vessels located on different sides of the river, and pour living water from a golden bowl into a vessel on one side of the river, noting its level. And then pour the same amount of water into an identical vessel on the other side
12	Volume. Measurement using a conditional measure	The stepmother ordered Cinderella to pour her and her daughters for tomorrow's breakfast such an amount of rice, oats, and millet (*show a reference*). But there are no more dishes of the same shape. How can we help Cinderella to get the right amount of rice, oats, and millet quickly as she wants to go to the ball?	Children need to use a small volume measure to measure how many times it will fit into the original volume of rice, oats, and millet

(continued)

Table 5.1 (continued)

No. of session	Contents	Example of the tasks	Solution
13	Volume. Assessment of the effectiveness of the volume measurement	The Fairy prepared magic pollen from flower petals, and she wants to hide the magic pollen in a pot for the winter. In her cellar, she found several pots (*show*). On the first pot it says "6 spoons," on the second—"8 spoons." The Fairy knows that she has prepared exactly 8 spoons of magic pollen But when she poured pollen into a pot with lettering "8 spoons," there was extra pollen left How could this happen?	Children should understand that the volume of spoons can be different and the amount of pollen in the pot depends on it
14	Volume. Standard volume measures	Each of the three gnomes poured 4 pots of living water into his jar. But look, there is different amounts of water in their jars. Who is right? The gnomes could not figure it out and quarreled. Let us find out who of them is right and help them not to quarrel anymore	Children should understand that the volume of the pot that each of the dwarfs used may be different
15	Comparison and measurement of different kinds of magnitude		

5.2 Evaluation of the Effectiveness of the Experimental Program

5.2.1 Methods

First, a diagnostic toolkit was developed for use with preschool children to assess the quality of their understanding of elementary mathematical representations of magnitudes and their relationships. This diagnostic technique included four types of tasks for each magnitude: length, area, and volume:

(1) two tasks each measuring the children's ability to compare objects; the children were asked to select an object the same size as another one. For example, find the rectangle with the same length as that shown in a drawing.
(2) two tasks each to measure the ability to use a measuring instrument correctly ("who measured correctly?"): to apply it so that there is no empty space between measurements, to use equal measuring instruments, etc.

(3) two tasks each for actual measuring of a magnitude with a conditional measuring instrument and recording the result with labels or a number ("how many times does the measuring instrument fit into this magnitude?").
(4) two tasks each to measure the children's understanding of how number depends on the measuring instrument used (the larger the measuring instrument, the fewer times it fits into the magnitude).

Two more assignments were included for creating sets ("what would be left over if such and such sets were used?"). The ability to put together complete sets was not specifically targeted in this study during the lessons, so we considered these tasks to be within the children's zone of proximal development (ZPD, Vygotsky, 2004). The tasks for making sets were assessed depending on the amount of assistance provided to the child by the tester and the correctness of the answer: the preschooler received 2 points for correctly solving the task independently, 1 point for solving the task correctly with the help of the tester's prompts; 0 points if the task was not solved or was solved incorrectly even with the help of an adult.

The formation of the concepts of length and area was scored on a scale from 0 to 10 points; of volume, from 0 to 7 points; and for tasks in the ZPD, from 0 to 4. The highest total possible score was 31 points. Diagnosis of mathematical concepts and skills was performed individually with each child.

Second, we used the level of development of the children's executive functions (EF) as the criterion for dividing the sample equally into experimental and control groups. We used the level of regulation to equalize groups, because based on the results of numerous studies, regulatory functions in older preschool age are closely related to mathematical skills (Bull & Scerif, 2001; Clements et al., 2016a, b). To measure the students' EF, we used the NEPSY-II subtests (Korkman et al., 2007) for visual memory ("Memory for Designs") and auditory working memory ("Sentence Repetition"); for inhibition and switching ("Naming and Inhibition", "Statue"); for cognitive flexibility ("The Dimensional Change Card Sort"); and for visual-spatial memory ("Schematization"). This allowed us to measure various components of the preschoolers' cognitive processes. The diagnosis of the children's EFs was reached based on two meetings with each child. As a supplement, nonverbal intelligence was diagnosed using Color Progressive Matrices by Raven (Raven & Kort, 2002).

The study was comprised of several stages. First, the cognitive processes of children were assessed using the NEPSY-II subtests along with Raven's colored matrices. After we evaluated their EFs, the children were divided into three subgroups by level of cognitive development (low, medium, and high) according to the results of cluster analysis (K-means clustering) performed in IBM SPSS Statistics 22. Before the experimental sessions began, a pre-test of mathematical concepts and skills was conducted using the authors' diagnostic tools.

Next, participants from each subgroup (those with low, medium, and high EF levels) were randomly assigned to the experimental and control groups, so that the ratio of participants in the groups was uniform. Then, 15 experimental sessions lasting 15–20 min were held in mini-groups of 3–4 children. The sessions occurred twice a week in the first half of the day in the groups at the kindergarten. The control group did not attend any special sessions. After the sessions, a post-test of mathematical conceptions and skills was performed, similar to the baseline diagnostics in the experimental and control groups.

5.2.2 Participants

The sample was comprised of 37 children aged 6–7 (mean age = 6.9) attending kindergarten preparatory groups, of whom 17 were boys (45.9%). No significant statistical differences were found in the level of mathematical abilities for boys and girls (Mann–Whitney U test, $p > 0.05$). Eighteen children attended experimental sessions; 19 were in control group.

5.2.3 Results

In order to assess the effectiveness of the program, we performed a nonparametric statistical analysis of the final scores of the pre- and post-tests for the experimental and control groups. The comparison showed significant differences in the total scores for diagnostics of mathematical ability between the results of the pre-test and post-test. The minimum overall post-test score in the control group was − 2 points out of the possible total of 31; this compared to a minimum score of 7.5 for the experimental group.

The participants in the experimental program showed a significantly greater increase in post-test scores than the children from the control group, in terms of the formation of concepts of length and area, the dependence of the number on the measuring method used; and the ability to select values and assemble "complex" sets (LSD, $p < 0.05$). Pairwise comparison of the increase in mean values showed that the children who participated in the experimental sessions showed a significantly greater increase in their total scores on the post-test as compared with the control group (LSD, $p < 0.05$).

5.2.4 Discussion

This research shows that the pre–number period is a key period for the further advancement of a child's mathematical development. From the CHAT point of view, the main goal of this period should be to prepare for the introduction of the basic concept for mathematics—the concept of number (Davydov, 1966; Solovieva et al., 2013). Considering number culturally and historically, we must highlight its main function: to display the relationship between the magnitude and the measuring tool by which this magnitude is measured. From this point of view, it can be assumed that the content of the number period should be familiarizing children with the concept of magnitude and how to measure it. Introducing magnitudes and actions with them allows us to start early algebra education, which has great relevance for mathematics education because it provides a special opportunity to foster algebraic reasoning and generality in our students' thinking (Lins & Kaput, 2004).

The data obtained in the study is consistent with other studies in this field which indicate that a measurement-based curriculum can stimulate the development of children's number knowledge and their sense of number (Cheeseman et al., 2018; Freiman & Fellus, 2021; Lins & Kaput, 2004). Moreover, it showed that acuity in understanding magnitudes correlate more significantly with overall mathematical achievements than number acuity up to the second grade (Park & Cho, 2017). Finally, a recent meta-analysis confirmed that magnitude processing is reliably associated with mathematical competence (counting, arithmetic, algebra, etc.) through a person's whole life (Schneider et al., 2017). This highlights the fundamental importance of pre-numerical learning for mathematical development.

In addition, our results demonstrate that it is necessary both to choose the right actions that will lead children to the assimilation of these concepts, and to choose adequate means for this age period. The program developed by us is based on the actions of selecting items by size through indirect comparison and the use of measurements. At the same time, we used symbolic means to emotionally involve children in the situation of the task. These symbolic means set up a problematic situation and key guidelines and ways for its solution, which become main points for mastering the concept of magnitude. It creates emotional meaningfulness of the action's purpose and facilitates the perception of the task (Treffers, 2012; Veraksa et al., 2014). This indicates the importance of choosing educational means appropriate to age-specific levels of development (Clements et al., 2016a, b).

5.3 Conclusion

From the point of view of CHAT, the content of mathematical education at school should differ fundamentally from reliance on the assimilation of mathematical concepts from preschool childhood; primarily it should stress the scientific nature of the proposed concepts and their systemic structure. It has been shown that the pre-numerical period of mastering mathematics can be considered a transitional period from preschool to school mathematical education. With the correct organization of the content of this period, the children's actions and the methods they use for their actions allow them to more effectively assimilate the basic concept for the subsequent study of numbers—the concept of magnitude.

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Chapter 6
Teaching-Orienteering Activity: Principles and Practices for the Organization of Mathematics Teaching

Elaine Sampaio Araujo

Abstract This work, developed in the ambit of an OBEDUC Project, presents pedagogical principles and practices that may be set up as basic properties that conduct the organization of mathematics teaching in the early years of Elementary Education. Defending the thesis that a certain theoretical-methodological basis guides the pedagogical process implies, necessarily, comprehending that the students' performance is attached to the kinds of teaching organization. In this perspective, we assume the Teaching-Orienteering Activity, based on the Cultural-Historical Theory, as a general kind of pedagogical organization. For that, we bring into discussion an activity in which this thesis is developed.

6.1 To Start the Conversation

Throughout our journey in teaching and research, a question accompanies us: Why do we do poorly in math? What is behind the numbers that indicate the low performance of Brazilian students in mathematics? Can these numbers subsidize referrals for a teaching proposal that has the participation of teachers? How does mathematics teaching take place in schools? These are all questions that could be asked by different segments of society: school, academia, public administration, private companies, financial sector, third sector, media, and whoever else wants to; after all, it is legitimate and right to talk about education. The media, for example, has the issue of (bad) education as a permanent agenda. However, if there is so much interest, if it seems to be a watchword in the discourses of different spheres of society, for what reasons does Brazilian education make little progress in terms of quality, particularly (but not only) concerning mathematics? There are many questions. In this study, we will seek

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to discuss them in a dimension that goes beyond the observation and presents itself as prospective, which meant considering each one of them as research questions.

Assuming a prospective dimension in relation to issues involving mathematics education was made possible by carrying out a research project entitled "*Educação Matemática nos Anos Iniciais do Ensino Fundamental: Princípios e Práticas da Organização do Ensino*" [Mathematics Education in the Early Years of Elementary School: Principles and Practices of Teaching Organization] (OBEDUC/PPOE), funded by CAPES, in the scope of *Programa Observatório da Educação* [Education Observatory Program].

Conducted in centers,[1] the research involved four Graduate Programs, in the period between December 2010 and June 2015, and had as its object of investigation the organization of mathematics teaching in the early years of Elementary School.

When considering the chosen theoretical-methodological foundation, the cultural-historical perspective, the research actions were based on two dimensions: the *collective* nature of knowledge production and the concept of *activity* as a formative unit of the subject. This meant considering the teaching activity, from the concept of Teaching-Orienteering Activity (Moura, 1997; Moura et al., 2010), as the premise and product of the pedagogical activity. The formative dynamics experienced in the collectivity, in the sense defended by Makarenko (1975, p. 101) as "um complexo de indivíduos que têm um objetivo determinado, estão organizados e possuem organismos coletivos" [a complex of individuals who have a determined objective, are organized, and have collective organisms], demanded the division of actions by centers in line with the research object. This work discusses actions developed in one of these centers, namely the one linked to FFCLRP/USP.

In this text, according to what was proposed in OBEDUC/PPOE, we initially present the object of investigation and the theoretical mode of apprehending it, which means understanding the principles of Teaching-Orienteering Activity. Once these principles have been discussed, we seek to demonstrate how they became practices, conforming to the form-content dialectical unity. In order to do that, we make use of teaching activities developed with children that, within the scope of the project, became activities for the centers to study and an effective possibility of understanding the thesis that a certain theoretical-methodological basis guides the pedagogical process.

[1] The centers that participated in this project were: Federal University of Goiás, under the coordination of Prof. Dr. Wellington L. Cedro; Federal University of Santa Maria, coordinated by Prof. Dr. Anemari R. L. V. Lopes; University of São Paulo, FFCLRP, under my coordination, and University of São Paulo, FE, under the coordination of Prof. Dr. Manoel Oriosvaldo de Moura, who is also the general coordinator of the project.

6.2 A Look Beyond the Data

When we take the results of external assessments as a starting point for the study of teaching organization, a first look at the performance of students in the area of mathematics raises concern. After all, they are data that reveal a difficulty for students to understand questions considered basic mathematical knowledge, such as problem situations in reality. This is what can be observed in the result of the Programme for International Student Assessment (PISA), under the responsibility of the Organisation for Economic Co-operation and Development (OECD), referring to the year 2012, in which Brazil, among 44 participating countries, occupies the 38th place. According to *Relatório Nacional PISA 2012: Resultados brasileiros* [PISA 2012 National Report: Brazilian results], available on the INEP website, the rate of Brazilian students who did not reach Level 2 of proficiency—a level that the OECD establishes as the minimum desirable—exceeds 60%. At this Level 2,

> [...] os estudantes são capazes de interpretar e reconhecer situações em contextos que não exigem mais do que inferência direta. São capazes de extrair informações relevantes de uma única fonte e de utilizar um modo simples de representação. Os estudantes situados neste nível conseguem empregar algoritmos, formular procedimentos ou convenções de nível básico. São capazes de raciocinar diretamente e de fazer interpretações literais dos resultados [students are able to interpret and recognize situations in contexts that require no more than direct inference. They are able to extract relevant information from a single source and use a simple representation mode. Students at this level can employ algorithms and formulate basic-level procedures or conventions. They are able to reason directly and make literal interpretations of results]. (Instituto Nacional de Estatística e Pesquisas Educacionais, 2014, p. 20)

Now, if the data indicate that the students' performance is precarious, they also indicate that something is worrying about mathematics teaching, concerning the curricular organization, that is, a learning problem is also a teaching problem. Although we are aware of the "vices of the products"[2] inherent to external assessments, particularly PISA, the results they present reveal something that we know, both through academic research and classroom experience: we do poorly in mathematics. This presents us with a challenge: to look at this reality considering its multiple determinations (Marx, 2011). This means that, behind the numbers of low student performance, we have several known determinants, such as: teachers' working conditions, initial teacher education, the commodification of teaching materials, public policies, the conception of education, the school's social function, the complexity of mathematical knowledge, pedagogical practices, among others. However, such determinants may be isolated for study. In this case, we took the determinant named "teaching organization", since all these issues are manifested in this unit of analysis.

> Com o termo unidade queremos nos referir a um produto de análise que, ao contrário dos elementos, conserva todas as propriedades básicas do todo, não podendo ser dividido sem

[2] In the area of consumer law, "vices of the products" are configured as errors from the origin. In the case of external assessments, this metaphor can be used as inaccuracies of principles and methods. This discussion is not the focus of this study, which may be seen in authors such as Ciavatta and Frigotto (2003), Figueiredo (2009), and Altmann (2002), indicated in the references of this work.

que as perca. A chave para a compreensão das propriedades da água são as moléculas e seu comportamento, e não seus elementos químicos. A verdadeira unidade da análise biológica é a célula viva, que possui as propriedades básicas do organismo vivo [By the term "unit" we mean a product of analysis that, unlike the elements, retains all the basic properties of the whole, not being able to be divided without losing them. The key to understanding the properties of water is the molecules and their behavior, not its chemical elements. The real unit of biological analysis is the living cell, which has the basic properties of the living organism]. (Vygotsky, 1991, p. 4)

A second look at this reality implies looking at the organization of mathematics teaching, not aiming to delimit which contents and procedures would be adequate, nor establishing desired levels of proficiency, but to understand which pedagogical principles and practices may be configured as basic properties that govern teaching organization. In this sense, what we seek here, in the light of the cultural-historical approach, when considering that the performance of students is linked to the modes of teaching organization, is to discuss how a certain theoretical-methodological basis guides the pedagogical process. And, here, old/new questions emerge: Who, when, what, how, and why to teach? We could immediately answer: everyone, all the time; mathematical knowledge as a cultural product; through teaching activities in which the social meaning of knowledge becomes personally meaningful to the student; for the new generations to appropriate human wealth and, in this process, for humanity to be formed in the individual. However, such an immediate response calls for an unfolding, in the sense of discussing it from the perspective of its genesis and development. That is, what the response movement is. Let us start by presenting the concept of activity, developed by the Cultural-Historical Theory.

6.3 Principles of Activity

Sem teoria não há ideia e sem ideia, não há teoria [Without theory, there is no idea and without idea, there is no theory]. (Kopnin, 1978, p. 323)

We do poorly in mathematics. This is a fact and, as Kopnin (1978) reminds us, in the theoretical construction, facts "are something reliable and irrefutable" (our translation), but they occupy the inferior position in the movement of theoretical production, which has the principles in the superior position. This understanding seems to be inverted in the debate about teaching organization in mathematics: principles, when not ignored, occupy a subordinate position in favor of facts, which are presented as the "garment" of principles, as the theoretical generalization itself. That is, the facts act in the theoretical construction as the generalization of principles, while the movement, according to Kopnin (1978, p. 325), is in the opposite direction:

No fundo, as teorias são constituídas de generalizações dos fatos, de abstrações. O princípio atua na teoria como essa generalização limite dos fatos, razão porque é abstrato, unilateral por natureza. O princípio mostra o grau de generalização a que se chegou em dada teoria [Basically, theories are made up of generalizations of facts, abstractions. The principle acts in theory as this limiting generalization of facts, which is why it is abstract, unilateral in nature. The principle shows the degree of generalization reached in a given theory].

A problem that such an inversion may generate is that the result of a process is mistaken for the cause. By taking the fact as a principle, the generalization is restricted to the empirical field and, in this case, the fact "we do poorly in mathematics" is only ratified by the performance, which, in turn, reveals that we are doing poorly. The identity between principle and fact creates restrictions for the phenomenon to be considered theoretically.

In this sense, when considering, as proposed by Kopnin (1978), that the purpose of the principle is that its generalization can be transferred to the interpretation of other phenomena, we make use of the principles of the Cultural-Historical Theory, particularly regarding the learning and development process, to discuss teaching organization through the Teaching-Orienteering Activity. The path we have chosen for this is based on the path developed in the research and begins with a synthesis of the concept of activity, in the Cultural-Historical Theory, adopting the Soviet authors who first developed this concept.

Studies in education in Cultural-Historical Theory, in the Brazilian context, in general, relate the concept of activity to Leontiev, attributing to him the development of Activity Theory. In fact, particularly with the publication of the work "The Development of Mind" in a Brazilian edition in the mid-seventies, Activity Theory became closely linked to the name of Alexis Leontiev. However, when we consider the context of production of Soviet psychology, especially between the 1920s and 50s, and the bibliographic production of that period, it is possible to observe an intense collaborative work between researchers, typical of a production mode that was intended to be socialist. Thus, although it is correct to relate Activity Theory to Leontiev, it is interesting to understand that other researchers have also studied it in form and content, as is the case of Rubinstein (1973) and even Vygotski;[3] even though this was not the main object for this author, the concept of activity was considered by him when studying the process of cultural psychological development of the child.

An example of this fact can be seen when Vygotski (1995) recovers from H. Jennings the biological concept of the animal activity system, understood as a system conditioned by the organs and in which the modes and forms of conduct are configured, to discuss that humans also have a system of activity that delimits the conduct. However, Vygotski emphasizes that the superiority of the human system lies in the possibility of using tools: "Su cerebro y su mano han extendido de manera infinita su sistema de actividad, es decir, el ámbito de alcanzables y posibles formas de conducta" [His brain and hand have infinitely extended his activity system, that is, the range of attainable and possible forms of conduct] (Vygotski, 1995, p. 37). This is why Vygotski, when discussing the child's activity system from the perspective of ontogenesis, sought to highlight the existence of organic and cultural development processes:

[3] The spelling of the name Vygotski will be considered according to the reference to which it is linked.

> Toda la peculiaridad del paso de un sistema de actividad (animal) a otro (humano) que realiza el niño, consiste en que un sistema no simplemente sustituye a otro, sino que ambos sistemas se desarrollan conjunta y simultáneamente: hecho que no tiene similitud ni en la historia del desarrollo de los animales, ni en la historia del desarrollo de la humanidad. El niño no pasa al nuevo sistema después de que el viejo sistema de actividad, condicionado orgánicamente, se haya desarrollado hasta el fin. El niño no llega a emplear las herramientas como el hombre primitivo, cuyo desarrollo orgánico se ha completado. El niño sobrepasa los límites del sistema de Jennings, cuando el propio sistema todavía se encuentra en su etapa inicial de desarrollo [The whole peculiarity of the transition from one activity system (animal) to another (human) that the child performs is in the fact that one system does not simply replace the other, but both systems develop together and simultaneously: a fact that has no similarity either in the history of the development of animals or in the history of the development of humankind. The child does not move on to the new system after the old, organically conditioned, activity system has developed to the end. The child does not use tools like primitive man, whose organic development is complete. The child goes beyond the limits of the Jennings system, when the system itself, however, is in its initial stage of development]. (Vygotski, 1995, p. 38).

In this sense, we may observe that Vygotski (1995), when borrowing the activity system theory from Jennings, did so considering the phylogenetic and ontogenetic processes of the human being. We could say that such an understanding was vital for the thesis about the social formation of the mind, developed by him in the study of the emergence and functioning of higher psychological functions.

In a similar perspective, Rubinstein declares that his study on psychic processes,

> [...] passou ao estudo da actividade na relação concreta com as condições da actividade efectiva. O estudo da psicologia da actividade, que efectivamente deriva sempre da pessoa como sujeito desta actividade, foi essencialmente um estudo da psicologia da personalidade dentro de sua actividade, isto é, das suas motivações (estímulos), fins e tarefas [moved on to the study of activity in concrete relation to the conditions of actual activity. The study of the psychology of activity, which in fact always derives from the person as the subject of this activity, was essentially a study of the psychology of personality within the activity, that is, of its motivations (stimuli), ends, and tasks]. (Rubinstein, 1973, p. 119)

We can consider that both Vygotsky and Rubinstein, when studying the higher psychic functions, come from a Marxist understanding of the historical formation of consciousness, defended by Karl Marx and Engels (2011), whose expression was marked in the work The German Ideology: "Não é a consciência que determina a vida, mas sim a vida que determina a consciência" [Life is not determined by consciousness, but consciousness by life]. This means to consider, roughly, that practical activity determines the mind. For Rubinstein (1965), the concept of activity is configured as a psychological category, while work would be a sociological category. Such distinction is important for the defense that both Rubinstein (1973) and Leontiev (1983) make about activity as an ontological category, which constitutes the being, whose analysis is carried out from the psychological perspective of human development.

Leontiev (1983), when discussing the correlation between internal and external activity, takes up Vygotski's ideas about the social formation of the mind and expresses that Vygotski understood the formation of higher psychological functions when analyzing human practical activity, from which he formulated a question that is central to the psychological understanding of human development: “El instrumento mediatiza la actividad que liga al hombre no sólo con el mundo de las cosas, sino también con otros hombres” [the instruments channel man’s activity, not only regarding the world of objects, but also the world of people] (Leontiev, 1983, p. 78).

In this sense, the concept of activity is related to the process of human development. In ontological development, according to the Activity Theory (Leontiev, 1983; Rubinstein, 1965, 1973), there are three dominant activities by which an individual is constituted as human: work and the activities of play and study that derive from it. Each of these activities is related to the place that the subject occupies in the system of social relations, which is why it is designated that the child’s main activity is to play, that of the young person is to study, and that of the adult is to work.

However, the concept of activity still has a broader understanding, referring to psychological processes, which, according to Leontiev (1978, p. 296), are: “caracterizados por uma meta a que o processo se dirige (seu objeto) coincidindo sempre com o objetivo que estimula o sujeito a executar esta atividade, isto é o motivo” [characterized by a goal to which the process is directed (its object) always coinciding with the objective that stimulates the subject to perform this activity, that is, the motive].

As a structure, the activity is characterized by two interdependent dimensions: execution and orientation. In the orientation dimension, we could consider the motive and the object toward which it is oriented. In such a way that motive and object are linked to a certain need. Actions and operations configure the executing dimension of the activity, in which objectives are related to actions and conditions to operations. In order to understand activity in its systemic relationships, it is important, initially, to understand the primary relationships, so that we can visualize the possibilities of movement. Assuming that, in order to constitute an activity, a coincidence between motive and object is necessary, if the activity loses its motive, it soon ceases to be an activity and becomes an action. However, there are cases in which the psychological process begins as action and, in its course, a motive arises and the action turns into activity; there is also the possibility that the action, when developed, changes to an operation, in a procedure to achieve a certain objective. This understanding of activity as a system, as proposed by Leontiev (1983, p. 89), is fundamental to discuss how this concept of activity may be considered in teaching organization because we start from the thesis that this is the unit of analysis that allows understanding the fact that “we do poorly in mathematics”. What we defend is that the concept of activity is, *par excellence*, the principle of Teaching-Orienteering Activity.

6.4 Activity Practices

In different works (Araujo, 2003; Lopes, 2009; Moraes & Moura, 2009; Moura, 1997, 2001; Moura et al., 2010) and especially the one developed in the OBEDUC/PPOE Project, there is an understanding of the Teaching-Orienteering Activity (TOA) as a training unit for the teacher and the student. We may understand this unit through the point that teacher and student are in activity. The teacher is in the work activity through teaching and the student is in the study activity. Both, as activities in the ontological dimension, must constitute the individual. That is, although teacher and student occupy different places in the system of relationships, TOA is configured as the human activity that mediates the relationship between these two subjects, so that its orienteering dimension intentionally leads to development: “La enseñanza debe orientarse no al ayer, sino al mañana del desarrollo infantil” [teaching must be oriented not to yesterday, but to the tomorrow of child development] (Vygotski, 2001, p. 242).

But which development? Skills and abilities? According to the Cultural-historical theory, it is about the development of human capacities that

> [...] não são dons inatos do indivíduo, mas produtos diretos das apropriações e objetivações efetivadas [...] o desenvolvimento de capacidades genuinamente humanas transcende o sentido utilitário e pragmático do conhecimento e da ação, pois demanda, o domínio de qualidades psíquicas amplas e estáveis, chamadas por Vygotsky de funções psicológicas superiores [are not innate gifts of the individual, but direct products of appropriations and objectifications carried out [...] the development of genuinely human capacities transcends the utilitarian and pragmatic sense of knowledge and action, as it demands the mastery of broad and stable psychic qualities, called superior psychological functions by Vygotsky]. (Martins, 2006, p. 36)

And then, we enter the realm of teaching. Which teaching? The one that orients learning and produces development. In this case, teaching in the dimension of Orienteering Activity has as content the teaching objects that bear the mark of the essential relationships of the objects of human activity, referring to certain knowledge. According to Nascimento (2014, p. 271), the teaching objects synthesize the general modes of action of the objects of human activity, which is why in the teaching object, human work is reproduced and, through it, if possible, the human-generic formation in each subject.

As a synthesis, we try to show the interdependence between Activity, Orientation, and Teaching in the Fig. 6.1.

In understanding the Teaching-Orienteering Activity as a theoretical-methodological basis for teaching organization (Mouta et al., 2010), and as a possibility to understand the fact that “we do poorly in mathematics”, there are two issues that we need to consider when we take the concept of activity as defended by Leontiev (1983). When considering the concept of activity as a system, it is possible to understand two dimensions that are interdependent. The orienteering dimension of the activity is related to the motives directed at the object, whereas the executing dimension of the activity is carried out through actions and operations.

Thus, for TOA, the first one, which concerns the orienteering dimension of activity, we could, in general terms, consider that the motive for TOA is to enable the social experience of humanity, objectified in culture, to become the experience of the subject, in such a way that the object of TOA is historically produced theoretical knowledge. However, what relates the motive to the object, from this perspective, is the social need for the formation of the human personality, that is, the pedagogical process as a formative process of the human personality. This is the need for teaching activity and study activity.

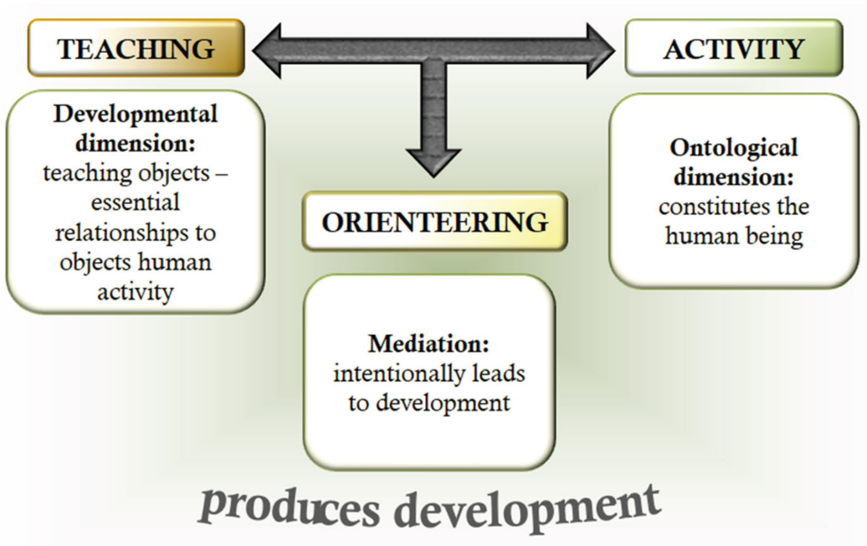

Fig. 6.1 Dimensions of the teaching-orienteering activity

The second issue to be considered relates to the executing dimension of the Teaching-Orienteering Activity. Here, the actions and operations come into play, so that the motive is fulfilled in the object. The actions, in general terms, are aimed at specific objectives that, in the case of school education, are identified with the appropriation of scientific concepts; and operations, for these actions to take place, go through modes of action that develop theoretical thinking. According to Moura, the Teaching-Orienteering Activity

> [...] tem uma necessidade: ensinar; tem ações: define o modo ou procedimentos de como colocar os conhecimentos em jogo no espaço educativo; e elege instrumentos auxiliares de ensino: os recursos metodológicos adequados a cada objetivo e ação (livro, giz, computador, ábaco, etc.). E, por fim, os processos de análise e síntese, ao longo da atividade, são momentos de avaliação permanente para quem ensina e aprende [has a need: to teach; has actions: defines the mode or procedures of how to put knowledge into play in the educational space; and elects auxiliary teaching instruments: the methodological resources suitable for each objective and action (book, chalk, computer, abacus, etc.). And, finally, the processes of analysis and synthesis, throughout the activity, are moments of permanent evaluation for those who teach and learn]. (Moura, 2001, p. 155)

How is the structure of the activity, in the orientation and execution dimensions, understood in the Teaching-Orienteering Activity? Initially, it is necessary to consider it from the perspective of teaching organization. So, we may resume the idea previously developed regarding the motive and object of the Teaching-Orienteering Activity, that is, the orienteering dimension of the activity. In this idea, the motive is to enable humanity's social experience to become the subject's experience; a motive oriented toward historically produced knowledge, which is constituted in the object of the activity. The unit of identity between motive and object is the social need for the formation of human personality.

How is this dimension fulfilled in the Teaching-Orienteering Activity? In pedagogical intentionality, when a certain theoretical plan orients the methodological plan. And, for that, it is necessary to understand the scientific knowledge to be taught, considering its conceptual nexuses. It is necessary to understand how the child learns and how humanity produced this concept and then define what are the founding concepts that allow "o social significativo tenha ao mesmo tempo significado pessoal" [the socially meaningful to have, at the same time, personal meaning] for the student (Rubinstein, 1973, p. 213). It is in the encounter between social and personal meaning that the motive and object of the teaching activity are substantiated. But how does it become practice? The activity is carried out through actions and modes of actions (operations), which is why the most apparent aspect of the activity is its executing dimension. However, it must be remembered that this is regulated by the orienteering dimension, which means that the execution is subordinated, at least initially, to the orienteering character of the activity.

From the perspective of Teaching-Orienteering Activity (Moura, 1996, 1997, 2000, 2001; Moura et al., 2010), the orienteering dimension takes place in the executing dimension, primarily by presenting students with a **learning trigger situation**. This action seeks to reproduce the essential relationships of the object of human activity in the teaching object, in form and content, so that the unity between the logical and the historical is reproduced in the teaching activity. According to Rubinstein (1973, p. 135), it means

> [...] elaborar adequadamente a matéria de aprendizagem para uma melhor apropriação, garantindo assim a apropriação de uma determinada matéria, de um determinado objecto. Este objecto possui a sua própria lógica objectiva, que não se pode impunemente descuidar. O "lógico", que progressivamente se vai formando no processo de evolução histórica do conhecimento, é também o comum que se relaciona tanto a evolução histórica do conhecimento como o processo do estudo entre si. E é aí que se assenta a sua unidade. Na evolução histórica do conhecimento percorreu-se um determinado caminho para elaborar este "lógico", cujo caminho reflecte a lógica do objecto de acordo com as condições concretas da evolução histórica [to properly elaborate the learning subject for a better appropriation, thus guaranteeing the appropriation of a certain subject, of a certain object. This object has its own objective logic, which cannot be neglected with impunity. The "logic", which is progressively formed in the process of the historical evolution of knowledge, is also the common that relates both the historical evolution of knowledge and the process of study to each other. And that is where its unity is. In the historical evolution of knowledge, a certain path was taken to elaborate this "logic", whose path reflects the logic of the object according to the concrete conditions of historical evolution].

In the learning trigger situation, there is the **presentation of the problem** that aims to highlight the social need of man to produce certain knowledge. The modes of action corresponding to such actions can be conformed to virtual stories,[4] games, and emergent everyday situations, for instance. Let us illustrate this with the virtual story

[4] Virtual Stories, in Moura's proposal, are configured as "problem situations posed by characters from children's stories, legends, or from the history of mathematics itself as triggers of the child's thinking in order to involve them in the production of the solution to the problem that is part of the

that has been constituted as a mode of action that allows the child, when understanding the problem of a certain character, moved by social need, to find the motive to seek a solution. The solution corresponds, in these circumstances, to the teaching object. Let us see a case. The teacher of the first year of elementary school intends to work with the Decimal Numeral System (DNS). What is its pedagogical intentionality? That children appropriate humanity's social experience is present in this mathematical knowledge. The object of human activity in this knowledge is linked to the movement of control, representation, and calculation of infinite quantities in a totally abstract way: with no visible signifier/signified relationship. In terms of teaching objects, this is equivalent to a numeral system that comes from the idea of "base", which corresponds to a certain number of signs, whose value changes depending on the position they occupy in a given order. The teaching object focuses on the conceptual nexuses of the DNS: the magnitude defined by the base principle. Understanding the decimal system, according to Vigotski (2010, p. 373),

> [...] leva à possibilidade de ação arbitrária nesse e em outro sistema. O critério de tomada de consciência reside na possibilidade de passagem para qualquer outro sistema, pois isto significa generalização do sistema decimal, formação de um conceito geral sobre os sistemas de cálculo [leads to the possibility of arbitrary action in this and other systems. The criterion of awareness lies in the possibility of moving to any other system, as this means generalization of the decimal system, the formation of a general concept about the calculation systems].

The triggering situation proposed by the teacher is based on the Virtual Story called "Shantal and Mira".[5] In this story, the characters are children from a distant country whose parents are shepherds and calculate the flock using an instrument that children do not know (in this case, the abacus). Desiring to know how it works, they ask their parents, who wisely ask them to discover it themselves and propose a challenge: they indicate on the abacus the number of animals that each family had. In order to perform the reading, it would be necessary to understand the operating logic of the instrument. A conflict arises, as each family had a different marking on the instrument and the children, sensorially, noticed that the marking, if they considered one mark for each animal, would not correspond to the number of the herd of each family. They really wanted to solve the problem and write a letter to the children of a school who attend first grade, who, when asked for help, are willing to solve the problem.

To what extent does this situation reproduce humanity's social experience? The abacus, as an instrument of calculation, probably has its origins in eastern civilizations, before the Christian era (Ifrah, 1998). In some of these civilizations, its use is still present, both in social activities and at school. A reality that is different in Western civilizations.[6] Its operating logic is based on the idea of position, with base ten, as in the Decimal Numeral System. Its creation was probably due to the need to

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context of the story. In this way, counting, performing calculations, and recording them can become a real necessity for them (Moura, 1996, p. 20, our translation).

[5] Version initially produced in *Oficina Pedagógica de Matemática* (USP) in 1995 and reintroduced in the OBEDUC/PPOE project in 2011.

[6] In America, the Incas developed something similar, the quipu, a cord with knots, in which they made the marking considering the position and nine units (IFRAH, 1998).

calculate taxes and large commercial operations. Transmitting this instrument from generation to generation is also a social need. The children, by identifying with the characters in the story, are motivated to discover how this instrument works, which is also unknown to them. The questions that instigate them are: how can they use it to count? Why do they need to find out? Because they need to write a letter in response to the girls who asked for help. The letter resource creates an appropriate cultural context for the first year of Elementary School, considering the social function of writing.

Once the problem is presented, through the learning trigger situation, the teaching activity is unfolded by the **discussion and understanding of the problem**: at this moment, children raise hypotheses, in such a way that language organizes thought and allows someone's understanding to be shared by someone else, as explained by Rubinstein:

> Devido ao carácter semântico da linguagem humana, podem, na comunicação consciente, com o próximo, designar-se os pensamentos e sentimentos e comunicá-los aos outros. Esta função semântica e significativa tão necessária para a comunicação formou-se dentro da própria pessoa, ou melhor, na comum actividade social dos homens, actividade que implica a comunicação prática real e ideal que se produz ao falar e que influencia mutuamente as duas partes. O homem fala para influir, se não directamente na conducta, ao menos no pensamento, nos sentimentos e na consciência dos demais seres [Due to the semantic character of human language, in conscious communication with others, thoughts and feelings can be designated and communicated to others. This semantic and significant function, so necessary for communication, was formed within the person, or rather, in the common social activity of men, an activity that implies the real and ideal practical communication that is produced when speaking and that mutually influences both parties. Man speaks to influence, if not directly on behavior, at least on thought, feelings, and conscience of other beings]. (Rubinstein, 1973, p. 18–20)

In the case of the virtual story about Shantal and Mira, children understand the problem: how to count using this instrument? How does it work? In this course of the activity, the children do not discover the operating logic, which reveals that, in fact, the logic of the abacus does not maintain a signifier-signified visual relationship, that is, it is not possible to understand its rules sensorially. But the children raise hypotheses: why do they not put all the marks on the same line? Do the marks have different values, depending on the lines? At this moment of **testing the hypotheses**, the first understandings of the concept emerge: the quantity defined by the principle of place value. So, in order to test the hypotheses, it is proposed that children use the abacus to keep the score in a bowling game; thus, they can notice some of its principles: multiplicative, positional, operating value of the number zero. The teacher intervenes revealing the functioning of the instrument. Here is a child's record of this activity, followed by its translation (Fig. 6.2).

[Diary of May 20, 1997]

I arrived at school and played with my friends before entering the classroom.

Coming in, we had an art education class. Coming back, with teacher Elaine, she read a story for us to understand a way to calculate everyone's total in the bowling

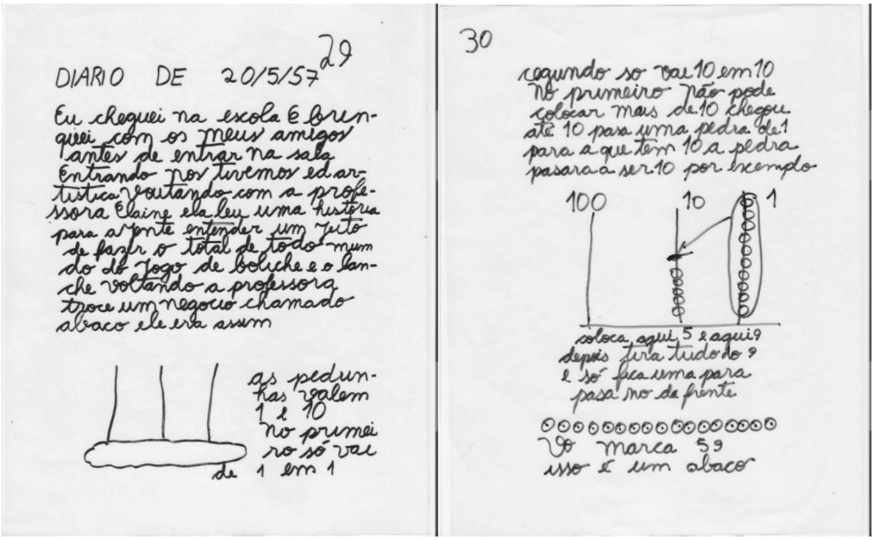


Fig. 6.2 Record from a child's notebook: explanation of how the abacus works (OBEDUC/PPOE collection)

game. Coming back from the break, the teacher brought a thing called "abacus". It was like this:

[drawing]

The beads are worth 1 and 10.

The first line only goes 1 by 1.

The second only goes 10 by 10.

You cannot put more than 10 in the first one. If you get to 10, you pass a bead from the line that is worth 1 to the line that is worth 10. The bead will be worth 10. For example:

[drawing]

You put 5 here and 9 here. Then, you remove everything from the 9 and leave just one bead to move to the next part.

[drawing]

I Will Make a 59.

This is an abacus.]

This child's record shows us the movement of their thought when explaining the abacus. We must consider that this is a six/seven-year-old child, in the first school

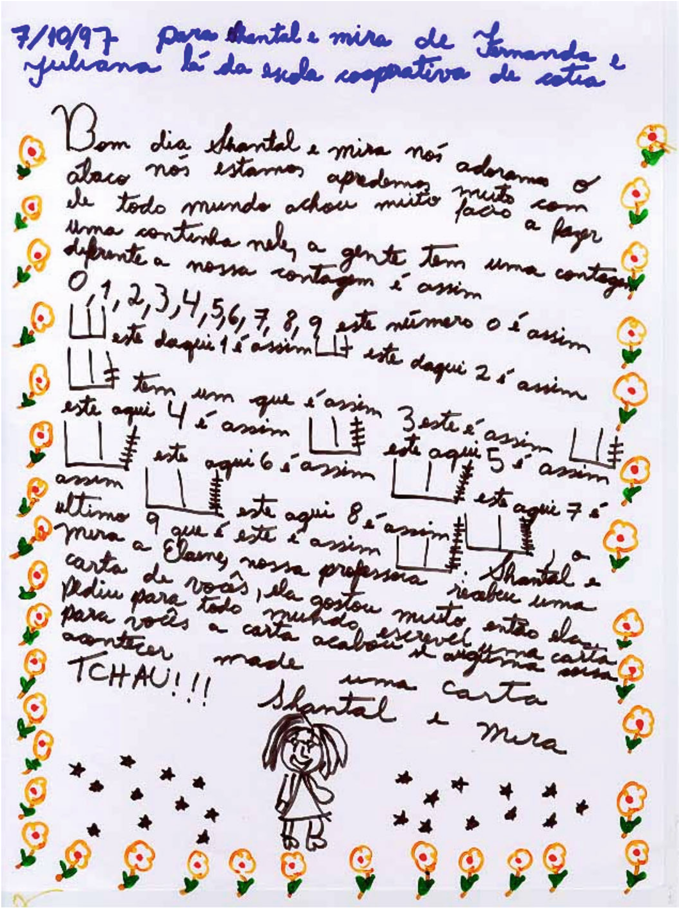


Fig. 6.3 Record of children replying to the Virtual Story (OBEDUC/PPOE collection)

year,[7] who seeks, through a written production that has a certain formal logic, to show how this counting instrument works, which is why there are some slips, gaps, typical of the thought organization process. However, the central idea of place value is explained. Thus, we arrive at the **definition of a solution** with the creation of a conceptual model that can be expressed, for example, with the writing of the letter in response to the problem presented in the Virtual Story by the characters Shantal and Mira.

Let us see how a pair of students replied, using the comparison between the Decimal Numeral System and the abacus (Fig. 6.3).

[October 7, 1997

[7] This material refers to the activity carried out in the period prior to the nine-year-long Elementary School.

To Shantal and Mira, from Fernanda e Juliana, from Escola Cooperativa de Cotia.

Good morning, Shantal e Mira. We loved the abacus. We are learning a lot from it. Everyone found it very easy to do math on it. We have a different way to count. Our counting is like this:

0, 1, 2, 3, 4, 5, 6, 7, 8, 9. This number 0 is like this [drawing]; this number 1 is like this [drawing]; this number is like this [drawing]; there is the number 3, which is like this [drawing]; this number 4 is like this [drawing]; this number 5 is like this [drawing]; this number 6 is like this [drawing]; this number 7 is like this [drawing]; this number 8 is like this [drawing]; the last one is the number 9, which is like this [drawing]. Shantal e Mira, Elaine, our teacher, got a letter from you. She liked it very much. So she asked everyone to write you a letter. The letter is over. If anything happens, send us a letter.

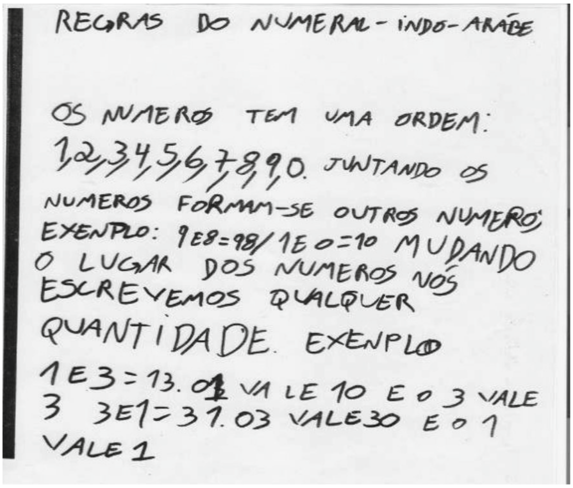
BYE!!! Shantal and Mira.

[drawing]]

We can observe in the record of these children some issues relevant to the study activity, which was the focus of OBEDUC/PPOE. There is a motive (Leontiev, 1983) to write the letter: to show the characters how the instrument works; and this becomes possible for the children in the first year comparing the abacus with the Decimal Numeral System, the object of study in this case. Although we may not observe all the conceptual links of the DNS in this record, some are manifested, such as the case of the operational value of zero and the use of the numerical sign. Another issue that deserves to be highlighted concerns the meaning that the activity aroused, which may be observed in the content and form of the letter, the drawings, the decoration, the availability to continue the conversation, the way of initiating the subject, saying that everyone loved the abacus, which brings us back to an issue pointed out by Leontiev (1983, p. 246):

> O que é que confere sentido ao que a criança estuda? O que faz com que ela compreenda a necessidade de estudar, os motivos reais de seu estudo? Conforme nosso postulado geral, a relação do objeto direto da ação relaciona-se ao motivo da atividade, dentro do qual está inserida dita relação, é o que nós denominamos sentido. Isto quer dizer, que o sentido que adquire para a criança o objeto de suas ações didáticas, o objeto de seu estudo, se determina pelos motivos de sua atividade didática [What gives meaning to what the child studies? What makes the child understand the need to study, the real reasons for their study? According to our general postulate, the relation of the direct object of the action is related to the motive of the activity, within which this relation is inserted. It is what we call meaning. This means that the meaning that the object of the didactic actions acquires for the child, the object of study, is determined by the motives of the didactic activity].

Thus, resuming the pedagogical intention of the teaching activity that had as its teaching object the conceptual nexuses of DNS—the magnitude defined by the base principle—from the **use of the solution**, it was possible, in different situations such as games, virtual story, and graphic activities, the regulation of action and the design of a mental plan. This can be seen, for instance, in the systematization that a pair of children carried out about the rules of the Decimal Numeral System, or the Hindu-Arabic system (Fig. 6.4).


REGRAS DO NUMERAL - INDO-ARÁBE

OS NUMEROS TEM UMA ORDEM:
1,2,3,4,5,6,7,8,9,0. JUNTANDO OS
NUMEROS FORMAM-SE OUTROS NUMEROS
EXENPLO: 9E8=98/1E0=10 MUDANDO
O LUGAR DOS NUMEROS NÓS
ESCREVEMOS QUALQUER
QUANTIDADE. EXENPLO
1E3=13. O1 VALE 10 E O 3 VALE
3 3E1=31. O3 VALE30 E O 1
VALE 1

Fig. 6.4 Synthesis of the decimal numeral system elaborated by the children (OBEDUC/PPOE collection)

[Indo-Arabic numeral rules

The numbers have an order: 1, 2, 3, 4, 5, 6, 7, 8, 9. Putting the numbers together, other numbers are formed; example: 9 and 8 = 98/1 and 0 = 10. By changing the place of numbers, we can write any quantity. Example: 1 and 3 = 13. 1 is worth 10 and 3 is worth 3. 3 and 1 = 31. 3 is worth 30 and 1 is worth 1.]

What to say about this synthesis? Did the object of human activity presents itself as an object of teaching activity? Could it be that the social experience of humanity, in creating a way to control the variation of quantities that was agile, efficient, safe, and easy, was appropriated by the new generations? We think so. And so, again, we use Leontiev to understand this relationship. And so, again, we make use of Leontiev to understand this relationship:

> [...] no basta que el niño asimile el significado del tema dado, sea teórico o práctico: es preciso que se conduzca como corresponde con respecto a lo que estudia, es preciso educar en él la actitud requerida. Sólo así los conocimientos que va adquiriendo serán para él conocimientos vivos, llegarán a ser auténticos "órganos de su individualidad" y, a su vez, definirán su actitud hacia el mundo [It is not enough to assimilate the meaning of the given object, regardless of what is done in theoretical or practical form, it is also necessary that an adequate relationship is reproduced in it concerning what is studied, it is necessary to educate it in this relationship. It is only satisfied if this condition, the acquired knowledge, becomes for him (child) living knowledge. They will be genuine "organs of his individuality" and, in turn, will determine his relationship with the world]. (Leontiev, 1983, p. 246)

We could end this text with the child's and Leontiev's words, such as the potentiality they represent. However, we would like to outline some considerations returning to the issue we initially proposed: to think about the organization of teaching through the Teaching-Orienteering Activity, as a way of theoretically answering the fact that "we do poorly in mathematics". In this way, we could say that the dimensions of orientation and execution of the Teaching-Orienteering Activity configure a general way of organizing teaching. This implies thinking of the teaching activity as the unit of the curriculum. From the teaching activity presented, we can visualize a synthesis of these dimensions when considering the curriculum practices:

- Trigger situation: it reproduces the essential relationships of the object of human activity in the teaching object, in form and content, considering the unity between the logical and the historical.
- Virtual story: humanity's social experience becomes the subject's personal experience.
- Presentation of the problem: it highlights the social need to produce the solution.
- Discussion and understanding of the problem: hypothesis raising and language organizing thought.
- Testing of hypothesis: first understandings of the essential relationships of the concept.
- Definition of a solution: elaboration of a conceptual model.
- Use of the solution: regulation of action, design of a mental plan.

Next, in the development of the defended thesis, that a certain theoretical-methodological basis guides the pedagogical process, and thus the student's performance is linked to the teaching organization modes, we seek to bring the final considerations with a conceptual tone to the continuity of the debate.

6.5 To Continue the Dialog

The discussion about the teaching activity as a curriculum unit, as defended in the OBEDUC/PPOE Project, has been oriented by three questions. The first one refers to the use of the term unit in the sense attributed by Vygotsky (2010), as a unit of analysis, which means for us that the teaching activity synthesizes the curriculum. Another issue concerns the understanding of the term "teaching activity". By "teaching activity", we are designating the pedagogical activities developed by the teacher. What we defend is that such activities are organized considering the theoretical-methodological bases of the Teaching-Orienteering Activity. That is, the TOA can be constituted as a teaching activity, but not every teaching activity is a TOA. The last issue is related to our understanding of curriculum; in this case, it is about thinking of the curriculum from the perspective of teaching organization.

Thus, for the curriculum to be carried out through teaching activities, from the perspective of Teaching-Orienteering Activity, it is necessary to consider some principles, which we have seen throughout this study and may be summarized in the Fig. 6.5.

In this sense, the course of the pedagogical activity (unity between teaching activity and study activity), from the perspective of the Teaching-Orienteering Activity, involves reproducing, in form and content, the object of human activity. This means that in the **learning trigger situation** we recover the historical-logical movement of the concept through a **triggering problem** whose solution goes through the process of collective creation, **through raising and testing hypotheses**, in which

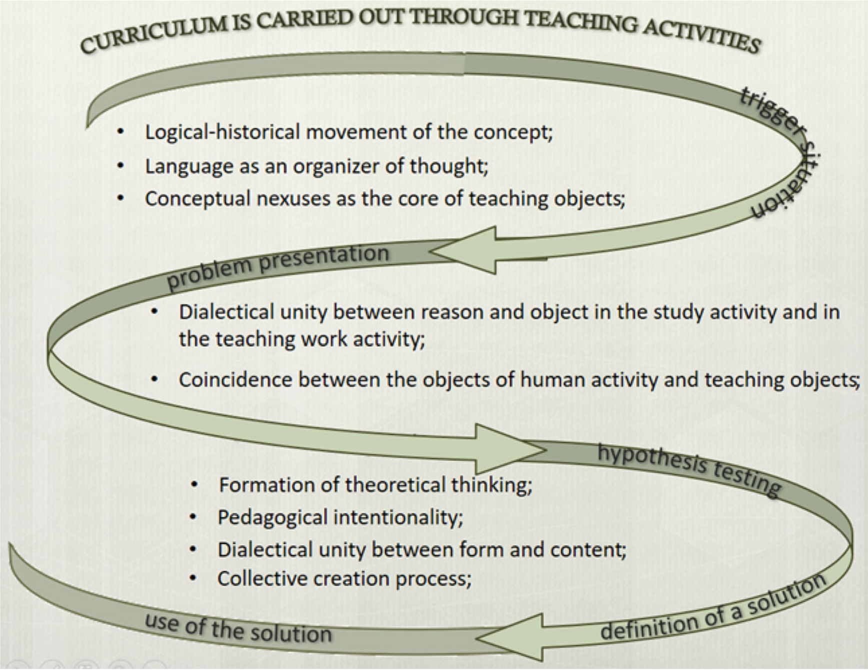


Fig. 6.5 Curriculum principles for the teaching activity

the conceptual nexuses present in the activity become evident, that allow us to reach a **solution** that may be used socially. This means reproducing, in form and content, the object of human activity, in its most elaborate dimension, in pedagogical activity.

In this way, when studying the fact that "we do poorly in mathematics", we outline principles and practices that make it possible to change this reality. It is possible to make another history; it is possible to "do well in mathematics". And the idea we presented to organize teaching was the Teaching-Orienteering Activity. Idea in the sense indicated by Kopnin, when stating that in the idea, in the field of scientific production,

> 1) estão expressas em forma concentrada as conquistas do conhecimento científico; 2) ela leva implícita a aspiração à realização prática, à sua corporificação material, à afirmação de si; 3) ela contém o conhecimento de si mesma, das vias e meios de sua objetivação, é um plano de ação do sujeito [1) the achievements of scientific knowledge are expressed in a concentrated way; 2) it implies the aspiration to practical achievement, to its material embodiment, to self-affirmation; 3) it contains the knowledge of itself, of the ways and means of its objectification, it is a subject's plan of action]. (Kopnin, 1978, p. 337)

This means thinking about teaching organization as an action in the field of scientific knowledge, in the understanding of education as a science. That is, we may have an idea of how to organize teaching, an idea in the field of scientific knowledge, which justifies defending a "generalized mode of action to organize teaching", which we call

Teaching-Orienteering Activity, and that, by containing the principles and practices, it is configured as a theoretical-methodological basis for both teaching activity and study activity.

In this sense, we can think of a curricular proposal that allows teachers and students to appropriate historically produced mathematical knowledge, in the dimension of the unity between form and content. This implies making another history in the teaching and study of mathematics, in which "doing poorly in mathematics" is a fact that has been overcome.

Within the scope of the OBEDUC/PPOE research project, this meant developing a curricular design based on the principles and practices of Teaching-Orienteering Activity. In order to do that, we carried out the study of the axes presented in "*Parâmetros Curriculares Nacionais*" [National Curriculum Parameters] (Ministério da Educação e Cultura, 1998), namely: Numbers and Operations, Measures and Quantities, Shape and Space, and Treatment of Information. This required the study of the logical-historical movement of the conceptual nexuses of each axis and the elaboration of a system of concepts, from which we developed a teaching proposal organized in the pedagogical material entitled "fascicle". Four fascicles were produced.[8] Their structure, in general, comprises the map of the system of the teaching objects of the axis (curricular design); the historical synthesis of the concept (the historical-logical movement of the concept), and triggering situations (Virtual Story) and their developments.

This experience was made possible by the structure and functioning of a program such as OBEDUC, which, by promoting the unity between teaching, research, and extension, enables the school to be present at the university as well as the university to be present at school. This produces a new quality in the relationship between the university and public school, which involves an understanding of the social function of the school as a social space in which knowledge is guaranteed as a good and right for all. Thus, the results of a project that integrates universities with different realities, researchers with mutual research objects and objectives, and schools, allow human wealth, present in mathematical knowledge, to be appropriated by all, that is, allow "doing well in mathematics" to be a reality for all.

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[8] The production of the fascicles involved all the participants in the project (undergraduate students, graduate students, elementary school teachers, and coordinators of the centers), in such a way that all the triggering situations were developed in the schools participating in the project, discussed, presented, and reformulated in the annual meetings of the project, which is why it results in a collective product.

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Chapter 7
Processes of Abstraction, Generalization, and Formation of Mathematical Concepts: A Teaching Experience with Future Teachers

Josélia Euzébio da Rosa

7.1 Presentation

This chapter presents some results from research on the mathematical thinking of future teachers of Pedagogy Faculty at a university localized in Southern Santa Catarina, Brazil. Along 2020s semester, an investigative teaching experience was carried out with thirty-four students of the fourth and sixth periods enrolled in the subject of Pedagogy Faculty called *Fundamentals and Methodologies of Mathematics for Elementary School Early Years,* which took place remotely because of the pandemic.

This is a teaching experience report, linked to understanding that by teaching, you learn, and through learning that one develops. However, it is not about any type of teaching, but about an organized teaching based on the contents and methods which allow developing theoretical thinking in future teachers from scientific knowledge appropriation.

Teaching experiment was arranged through problem development, within Learning Triggering Situations context. For the analysis, episodes that indicate the movement of thought traversed by academics from the general to the particular and singular were chosen.

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7.2 Theoretical Context

This chapter is a result from an investigation whose purpose is at reflecting on the limits of the way of organizing empirical teaching and its overcoming possibilities based on two of the Historical-Cultural Theory developments: Teaching Guidance Activity, also called teaching Guiding Activity, which has already been covered in the previous chapter, and Developmental Teaching Theory.

In Basic Education, it is common to come across mathematical development at an empirical level (Eriksson & Eriksson, 2021; Eriksson & Sumpter, 2021; Rosa & Fontes, 2022; Venkat et al., 2021). In empirical thinking context, mathematical concepts are approached without the appropriate mathematics context, through a linear and fragmented sequence, and from a direct and superficial relationship between objects and phenomena with symbols and operations taken as a sequence of procedures to be carried out, without understanding what generates such procedures and with what purpose. Theoretical thinking, on the other hand, is further than appearance, and enters the concepts essence (Davídov, 1988).

To develop theoretical thinking in schoolchildren, a Russian researcher group with D. B. Elkonin and V. V. Davydov, among others, systematized the Developmental Teaching Theory, and the Brazilian professor Manoel Oriosvaldo de Moura proposed the Teaching Guidance Activity.

Teaching Guidance Activity constitutes a general way of teaching organization. Teaching main content is theoretical knowledge, and therefore, its object is constitution of students' theoretical thinking in the knowledge appropriation movement (abstraction, generalization, and concept). Teaching Guidance Activity is a teacher and student training unit, since the teacher, when organizing actions, aims to teach, requalifies knowledge (Moura et al., 2016).

As a theoretical-methodological proposal, Teaching Guidance Activity must contain historical synthesis of the concept, teaching resources, analysis, and collective synthesis in its structure during performance of situations which trigger learning (Moura, 1996).

Learning triggering situation consists of a proposal organized by the teacher who, based on his or her teaching objectives, leads the conceptual movement, to be appropriated by the students, through a learning problem (Moura et al., 2016). Teaching Guidance Activity, systematized within the Brazilian context, communes with the task of the contemporary school, taken by precursors of Developmental Teaching Theory.

> [...] the task of the contemporary school does not consist in giving children a sum of known facts but teaching them how to orient themselves independently in scientific information and in any other one. This means that the school must teach students how to think, in other words, actively developing in them the foundations of contemporary thought, and to do so, organizing teaching that encourages development is necessary (Davídov, 1988, p. 3).

When taking as ours the task of the school proposed by Davídov (1988), and driven by the mentioned theoretical assumptions, we ask: How to organize mathematics teaching with potential to promote the scientific concepts learning and development of theoretical thinking?

In searching for answers process for the previous question, we take the movement of mathematical thought of students of a Pedagogy Faculty, future teachers, as research subjects to investigate, at theoretical level.

Historical-dialectical materialist method sustains our research, teaching, and extension actions. One of the main characteristics of this method is the premise that investigated phenomenon must be considered in its entirety, in inseparability between theory and practice.

Research methodology used was teaching experience (Davídov, 1988). Such a methodology is linked to understand that by teaching, you learn, and through learning that one develops. However, this is not just any teaching, but an organized teaching based on contents and methods which potentiate development of theoretical thinking in students (children, adolescents, young people, and adults) from the scientific knowledge appropriation.

This research methodology allows the researcher investigating the movement of students' thinking in the teaching and learning process. According to Davídov (1988), teaching experience is characterized by researcher's active intervention in the processes he investigates. Therefore, it is essentially different from the verification experiment, which highlights only the state already formed and presented by students.

This research proposal "[…] appears as an experimental education and teaching methodology that drives development" (Davídov, 1988, p. 196). Performing teaching experience sought to project and model the essential relationship of concepts in learning process, as shown below.

7.3 Investigation Experimental Context

Along 2020s semester, a teaching experience was carried out with thirty-four students of the fourth and sixth periods enrolled in the subject *Fundamentals and Methodologies of Mathematics for Elementary School Early Years* in Pedagogy Faculty. On the first day class, the project was presented, and all academics agreed to collaborate with the research.

The group was very diverse from an age viewpoint, with students from 18 to 48 years old. At the beginning of the semester, most academics were already starting their teaching careers, through paid internships or teaching assistants to students with special needs. Most of the group reported they experienced difficulties with mathematics in Basic Education, and that these difficulties turned, over the schooling, into an aversion to this subject. Such aversion provoked anxiety and concern in some students when the mathematics learning subjects started in the Pedagogy Faculty. Over the semester, insecurities caused by previous negative experiences were gradually and partially cooled.

The subject syllabus presents three broader goals: Expanding students' need for learn; developing theoretical thinking, intellectual autonomy, and creative activity;

and promoting development of different understanding, reflection, and analysis modes. Within these broader goals context, the fundamentals of teaching organization are studied to guide the learning process in Elementary School Early Years. Also, there are development of: (1) foundations of scientific mathematical thinking; (2) Mathematics concepts as universal language; (3) arithmetic, algebraic, and geometric modeling of phenomena and processes; (4) logical thinking; (5) algorithmic culture; (6) spatial imagination; (7) ability to build arguments; (8) asking questions; (9) ability to analyze situations in terms of mathematical properties; (10) performing quantitative and spatial relationships between objects to build an algorithm and find information; (11) ability to solve problems through universal problem solving models; (12) ability to plan; and (13) interest in Mathematics Education and desire to develop mathematical knowledge in teaching at a theoretical level.

Intended skills, according to the teaching plan, are: learning autonomously and thinking theoretically; need for continuous self-development; creative personality; organization of the own study activity, by defining learning goals and modes of study action; ethical commitment to the process of learning scientific knowledge and developing students' theoretical thinking; critical reading of the teaching organization traditionally developed in Brazil; and recognition of possibilities of overcoming the theoretical way of organizing empirical teaching in the study activity context.

Classes were held on Tuesdays, from 7:15 to 10:30 pm, through Zoom platform, because of the pandemic caused the SARS-CoV-2. The classes were recorded by the Zoom application and made available weekly by the head teacher, who is this chapter's author, to the students. There were fifteen meetings in all. To discuss in this chapter, five episodes were selected, four from the tenth and one episode from the eleventh class.

From methodological viewpoint, classes were held through dialogic study to enable interaction among learning participants. Thereunto, there was a suggestion that the cameras of the learning subjects be kept open during classes, so that everyone could interact with everyone else. Microphones should only be opened during oral manifestations.

During the classes, mathematical conceptual systems were developed from Learning Triggering Situations, in line with theoretical foundations arising from Developmental Teaching Theory and Teaching Guidance Activity. Conceptual systems are understood as those concepts which are interconnected with each other, either in an inverse operation form, or in the common operational core, such as addition and multiplication, for example.

Therefore, in the classes, students not only reflected on theoretical foundations related to the way of organizing teaching, but these foundations were experienced. They were embodied, objectified in the collective development of the problem of each Learning Triggering Situation.

Educational process organization took place through moments of previous studies, carried out individually or in groups, and dialog of collective studies during classes. Pre-reading material was posted on the class academic system in advance, so that everyone could carry out a preliminary study individually. The classes were developed with everyone and by everyone, through a joint action. Wherefore, in the collective and for the collective, academics were protagonists of their own learning and development process. In addition, they were co-responsible for the learning and development process of their colleagues through collaborative action.

During the search for problems solution, students were instigated to pose questions, formulate different hypotheses for solutions, and confirm or refute such hypotheses.

7.4 From Totality to Analysis Clipping

Teaching experience was arranged through problems development in Learning Triggering Situations context and previous study of texts that would support the reflections to be carried out in class. Mathematical concepts such as number, addition, subtraction, multiplication, division, among others, were not approached separately, but interconnected in their respective conceptual systems (Vigotski, 2001). Neither moment nor part of the class was separated to talk about theoretical foundations of teaching organization, nor to teach mathematical concepts. Reflections on theoretical foundations previously studied by the students emerged during classes from their objectification in the organizing way of teaching mathematical concepts. In addition to developing problems presented in the Triggering Learning Situations, students also crafted their own as well because it is understood that teacher training involves not only learning the process of solving existing problems, but also proposing new ones.

During the semester, Learning Triggering Situations were developed from the study of discrete and continuous magnitudes, such as length, monetary value, area, volume, time, angle, among others (Rosa & Albino, 2021; Rosa & Antunes, 2021; Rosa & Becker, 2021; Rosa et al., 2021; Rosa & Marcelo, 2022; Rosa & Santos, 2020).

Given the impossibility of covering the entire teaching experience within the limits of this chapter, to provide an example, a Triggering Learning Situation was chosen. It was carried out with students along two classes which started at 7:15 pm and ended at 10:30 pm. They were the tenth and eleventh classes. Reflections performed in these two meetings reflect and synthesize the movement of thought covered by the academics in previous classes.

Thereby, five episodes extracted from reflections regarding the movement of mathematical thinking followed by academics in the process of searching for a solution to a problem are presented, as follows.

Teaching objective was reflecting with future teachers on development possibilities of a mathematics conceptual system with children in the context of a learning triggering situation.

Data source consists of transcription of students' statements from the Zoom recording. All the students agreed to participate in the research. To preserve their

Table 7.1 Learning triggering situation

Liandra's request for help to students of the Pedagogy Faculty
Hello, future teachers, how are you? My name is Liandra, and I am your teacher's goddaughter. I am eight Years old, and I am enrolled in the second grade She told me you are hard students and very helpful. So, I asked her to send you this letter Now, during the pandemic, I am not going to school. My mother is who helps me with my homework Mila[a], Physical Education teacher has sent a message to my mother with a task on long jump. The message explains that I should make a mark in the sand, run to the mark, and then jump. She challenged me to jump as far as possible I jumped very high. My feet went so far that I had to support myself with my hands to keep from hitting my butt on the ground. My jump was big, but my mom said it wasn't that big. We didn't reach a consensus and I must report my jump measurement to teacher Mila[a] Could you help me understand who is correct, me or my mother? How should I proceed to know the correct measurement of my jump? If you find out, please send me a letter explaining how I should proceed to correctly measure my jump Thanks in advance, Liandra

Note From Rosa et al. (2021)
[a] In Brazil, teachers are usually called by their first name

identities, the letter A means academic, followed by a number to reference them. When the speech belongs to the research professor, it is indicated by PP.

7.5 Episodes Analyzed

Next, five episodes extracted from the tenth and eleventh classes are presented, from a teaching experience carried out with students of the fourth and sixth periods of a Pedagogy Faculty, in the subject *Fundamentals and Methodologies of Mathematics for Elementary School Early Years I*. Episodes showed by speeches by academics during the reflections inherent in the process of searching for a solution to the triggering problem of the Triggering Situation of learning below[1] (Table 7.1).

Triggering problem of the Learning Triggering Situation entitled *Liandra's request for help to students of the Pedagogy Faculty* consists of the following question: How should I proceed to know the correct measurement of my jump? It is important highlight that the answer to this question goes through the general solution mode, valid for any measure of jump. From the search process for a solution to the problem experienced by Liandra, five episodes were extracted for analysis, as follows:

[1] Learning Triggering Situation *Liandra's request for help to students of the Pedagogy Faculty* was elaborated based on the Triggering Situation entitled *Julia's letter to the first grade* (Rocha et al., 2019, p. 28).

7.6 Episode 1—Revealing Elements that Make up the General Solution Procedure

If we were in the classroom, not in the pandemic times, we would have taken the students to perform long jumps outside, such as Liandra's teacher suggested. However, facing the pandemic context, the students were suggested to carry out the jump in their own places. Based on this body movement performed during the jump, students' thinking was guided to reveal the essence of conceptual system to be studied, as described below:

Autonomously and collectively, students raised some assumptions about what would have caused the problem (Table 7.2) and concluded on the need for a common measurement unit for mother and daughter to measure using a string. Thereby, they revealed two important elements of the solution procedure to the triggering problem, the distance length to be measured and the string length as a unit and measuring instrument. But the values are unknown, how to represent them? The answer to this question was as follows (Table 7.3).

Representing unknown measures through letter stems from the need to communicate Liandra, the solution procedure under development. The group made the option by the letters t and u (Fig. 7.1).

Figure 7.1 shows the first episode starting point, in which the jump distance length (t) is represented, and the basic measure unit (u). With these two measures, the aim was at revealing the genesis of the concept of number through geometric meanings (segments and arcs) and literal ones (t and u). Arithmetic significance is revealed in the second episode, during the measuring operation.

Table 7.2 Assumptions on the cause of Liandra's problem

A_3: They might have used different units of measurement, maybe her mother measured using her own steps. Because they are two different measurement units, a child's step and an adult one, right?
A_5: She mentioned in the text she has jumped very high. Jumping too high doesn't mean that the distance covered is too great
A_4: It depends on how she has put her feet down. Also, how she put her hands on the ground and the place that the marking was considered, where she put her hands or where she fell
A_5: There is no way to calculate it. You must establish a fixed unit of measurement
A_{16}: Using an object
A_{20}: Maybe a string

Note From Rosa et al. (2021)

Table 7.3 Representation of unknown values through letters

A_{11}: You must assign a letter because we don't know the size
A_{21}: It can be any letter

Note From Rosa et al. (2021)

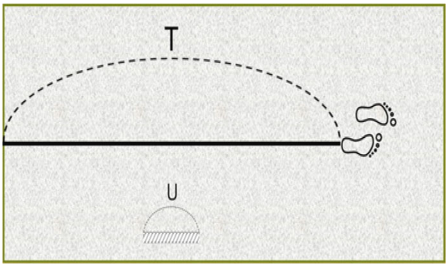


Fig. 7.1 Two elements that compose the problem. *Note* From Rosa et al. (2021)

7.7 Episode 2—Modeling of Basic Relation of General Solution Procedure

In the second episode, the essential relationship was modeled in object, graphic, and literal forms. Data revealed and abstracted in the first episode enabled modeling the essential relation of the number concept by the need for measurement (Table 7.4).

From the suggestion by A_5, "Introducing the measurement unit *u* in the course", the measurement was performed by overlapping the string on the length of the jump distance, and a picture was shown through a screen shared with the group by Zoom. The unit was marked with a dash until it reached the point where Liandra's feet touched the ground. During the measurement, the numbers were synchronously registered on the number line. Therefore, when the first measurement was carried out, the number was registered on the number line, and so on. This process was concluded by the construction of Fig. 7.2.

Applying process of the measurement unit on the magnitude has geometrical character. The number of times the unit fits in the magnitude translates the arithmetic content, which arises from the algebraic relationship between quantities. Numerical property of the magnitude varies according to the variation of the measurement unit (Caraça, 1951).

From the measurement (Fig. 7.2), the group concluded the measurement unit (u) repeated itself six times in the magnitude measured (t). Behold, the third element

Table 7.4 Reveling essential relation from the need for measurement

PP: […] What is the arithmetic value of t?
A_5: You must use the measurement unit that we have there, which is the string. You must see how many times it will repeat itself
A_{18}: The number of times u will repeat itself within T
A_{17}: We must start the measurement from the starting point
PP: And how are we going to represent the starting point on the number line?
A_3: Zero represents the starting point
PP: And how do we carry out the measurement?
A_5: Introducing the measurement unit u in the course
A_2: In fact, how many times does the string fits

Note From Rosa et al. (2021)

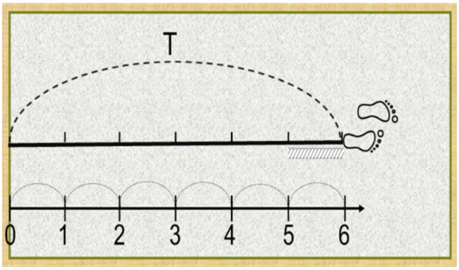


Fig. 7.2 Measurement. *Note* From Rosa et al. (2021)

of the essential relationship of the number concept arose: the number of times a measurement unit fits into a quantity to measure. These three elements compose the essential relationship from which the natural and rational numbers originate (Rosa, 2012). Interconnection of these elements is made explicit in the graphic modeling (Fig. 7.3).

Based on the graphic modeling (Fig. 7.3), we question: how to algebraically represent the relationship between the measures t and u? Students proposed the following models (Fig. 7.4).

When concluded the previous modeling (Fig. 7.4), A_{14} asked a question (Table 7.5).

The question by A_{14} promoted the continuity of the modeling process toward maximum abstraction, or as suggested by A_9, toward the universal model (Fig. 7.5).

In Fig. 7.4, despite using letters, it was still dealing with specific modeling, valid for cases in which the measurement unit fits six times the magnitude measured. The question by A_{14} triggered the continuity of the abstraction and generalization process toward the universal model (Fig. 7.5), and therefore, valid for measuring any magnitude from any unit measurement, regardless of the number of times the measurement unit fits into the magnitude measured. Next, based on the assumption of a Liandra's new need, the need for transform the model was introduced.

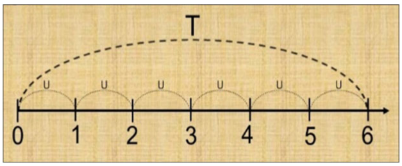


Fig. 7.3 Graphic modeling. *Note* From Rosa et al. (2021)

$$t = 6u \qquad \frac{t}{u} = 6$$

$$\frac{t}{6} = u \qquad u = \frac{1}{6}t$$

Fig. 7.4 Literal modeling. *Note* From Rosa et al. (2021)

Table 7.5 Need for modeling process continuity

A_{14}: We are already in the literal [representation]. But would it be, for now, the junction of the literal with numeric one? [Regarding the numbers one and six presented in the models]
PP: Is it possible to advance in the modeling process through yet another abstraction movement?
A_5: It is possible to put a letter representing the number six. Everything can be represented by letters, then just replace the letters
A_9: Then it would be the universal model

Note From Rosa et al. (2021)

Fig. 7.5 Maximum abstraction of essential relationship. *Note* From Rosa et al. (2021)

$$t = pu \qquad \frac{t}{u} = p$$

$$\frac{t}{p} = u \qquad u = \frac{n}{p}t$$

7.8 Episode 3—Deduction of a New Relationship

In the third episode, transformation of the essential relation model for the study of its properties was performed to reveal interconnections of the conceptual system under study. To trigger the model transformation process, the class was informed that Liandra and her mother used a common measurement unit. Then, the hypothesis that mother and daughter would have adopted different measurement units, as supposed in the first episode, was refuted. Hence, new questions were asked by the researcher (Table 7.6).

New hypotheses have emerged, and now the finding of the cause of the conflict between Liandra and her mother opened the way to study the relationship of inequality and difference between the measurement results.

Some students have suggested that to solve the problem Liandra could consider his mother's measurement, how far the hands touched the ground. Others indicated the possibility of considering the measurement carried out by Liandra, measuring

Table 7.6 Need for model transformation

PP: What could have happened? Liandra thought she had jumped too far, but her mother didn't
A_{21}: Maybe one considered the hands on the ground and the other as far as the feet touched
A_{22}: In the long jump, no matter where the feet touch if some other part of the body touches first. Even the feet touching far, if the hands touched the ground behind the body, what will be measured is from the point where it jumped to the first mark. In this case it was where her hands touched the ground
A_{16}: The rules must be considered [long jump ones]
A_2: If you measure by the feet, it is larger, and by the hands, it will be smaller
A_5: The mother considered the hand

Note From Rosa et al. (2021)

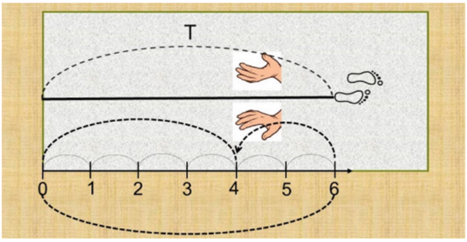


Fig. 7.6 Subtraction operation on the number line. *Note* From Rosa et al. (2021)

from where the feet touched the sand to where the hands touched, to subtract that part from the whole. As example, if Liandra concluded that the jump measured six units, it would only be necessary to subtract the difference. The suggestion was accepted, and Fig. 7.6 shows the procedures:

Figure 7.6 represents operation $6 - 2 = 4$ on the number line: when subtracting the invalid part (2), results the valid part of the jump (4), according to Table 7.7.

Reveling subtraction as an inverse movement to that performed by Liandra in the jump coincides with theoretical meaning of that concept in natural numbers context. Subtraction is the operation performed when the whole is known and one of the parts is unknown (Matos, 2017). In natural numbers context, subtract a unit occurs by shifting to the left on the number line (Silveira, 2015).

When asked "what would be the inverse operation of that represented on the number line", students answered addition, and A_9 detailed: "four plus two equals six". They were asked how they could represent the relationship between the parts and the whole in literal form. They suggested representing the parts and the whole by "different letters" (A_{20}), and "they will represent different things" (A_{22}). After defining the letters that would be used, students developed the following literal models (Fig. 7.7).

Table 7.7 Internal movement of elements that compose subtraction

A_{17}: It is subtracting two from six
A_2: It's shortening the length
A_9: It is noticeable that direction has changed, right? It's decreasing

Note From Rosa et al. (2021)

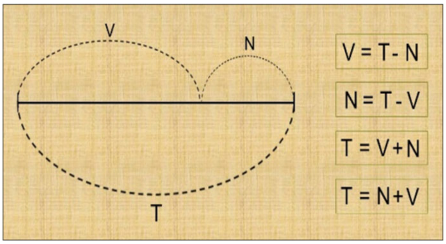


Fig. 7.7 Literal modeling from the conceptual core constituted by the whole-parts relationship. *Note* From Rosa et al. (2021)

Model transformation process showed by Fig. 7.7 subsidized the reflection on some addition and subtraction properties. These abstractions represent the relationship between the T, V, and N measures in addition and subtraction operations context. In other words, from the cell geometrically expressed through segment and arcs, the abstract models are deduced, which in summary reflect the whole-parts relationship.

When returning to Liandra's problem, students concluded that it would be solved by subtracting the invalid part of the jump from the distance she jumped, regardless how much she has jumped and the size of invalid part. It is therefore a general solution procedure.

In summary, in the third episode, reflections were conducted until reaching the abstraction and generalization of the general model for solving problems involving addition and subtraction operations (Matos, 2017): if the two parts are known, the operation to be performed is addition. When the whole and one of the parts are known, it is a subtraction. This is the essential constitutive relationship of the nuclear cell that connects addition and subtraction in dialectical unity.

Then, new elements of conceptual system under study were revealed in the third episode, such as parts and whole, from elements revealed in the first action and modeled in the second one. Therefore, there was an overcoming due to the incorporation of previous actions. Genesis of multiplication and division concepts was revealed along with the number, in dialectical unity as well, and only later the genesis of addition and subtraction concepts was introduced.

7.9 Episode 4—Summary of Two Previous Relations

Are the reflections carried out so far enough to guide Liandra in the search for a solution to her problem? And if the string does not fit an entire number of times in the valid part, how should Liandra proceed? When answering these questions, A9 suggested subdividing the string. It was divided into half (Fig. 7.8).

Subdivision of measurement basic unit into two equal parts was not sufficient to perform the measurement accurately (Fig. 7.8); the result was a little more than three measurement units. Next suggestion was to subdivide the unit into three equal parts, which enabled the measurement, as shown in Fig. 7.9:

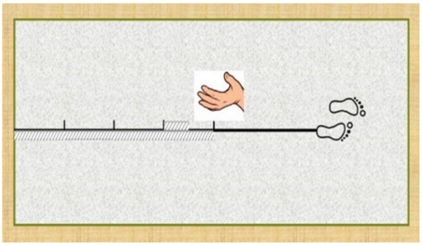


Fig. 7.8 Revealing intermediate measurement unit. *Note* From Rosa et al. (2021)

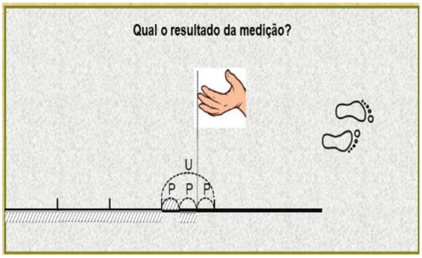


Fig. 7.9 Measurement using basic and intermediate measurement units. *Note* From Rosa et al. (2021)

By dividing the unit into three equal parts, two of them fit in the valid part of the jump that remained to be measured, that is, two thirds. Then, from the procedure revealed in the second episode, a new measurement unit was deduced, the intermediate one, which in the example under analysis, corresponds to a part of the basic measurement (u). The group and the researcher decided that each subdivision of the basic measurement unit would be represented by the letter p and there was a question, on how would be algebraic representation of the relationship between p and u (Fig. 7.10).

This movement was important to guide students' thinking beyond body movement and object representation exposed in the images of the situation, so that they entered the interconnection of these elements. Next, there was another question (Table 7.8).

Students were questioned the reason why eleven thirds, and the answer culminated Fig. 7.11.

The tenth lesson was completed by explaining the reason why the answer was eleven thirds through the arithmetic expression (Fig. 7.11).

$$u = 3p \qquad \frac{u}{p} = 3$$

$$\frac{u}{3} = p \qquad p = \frac{1}{3}u$$

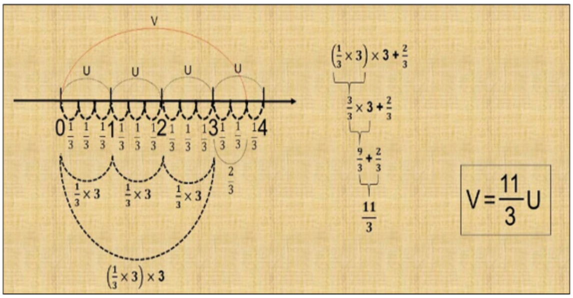
Fig. 7.10 Revealing internal connection of basic measurement unit and intermediate one. *Note* From Rosa et al. (2021)

Table 7.8 Implementation of the general solution procedure

A_{17}: What is the measurement result? [Fig. 7.9]
A_9: Three whole parts plus two thirds
A_5: $3u + 2p$
A_{24}: Three plus two thirds
A_9: I could say it's eleven thirds

Note From Rosa et al. (2021)

Fig. 7.11 Implementation of the general solution procedure. *Note* From Rosa et al. (2021)



7.10 Episode 5—Answer Letter

Next class started with groups elaborating the answer letter to Liandra. The activity took one hour. Then each group presented its version of the letter. Answer letter elaborated by the first group that presented its activity was chosen to show in this chapter because they finished first and they have no interference from other groups' presentation, as given in Table 7.9.

Development of the Triggering Learning Situation was concluded by elaboration an answer letter to the problem, entitled Liandra's request for help to students of the Pedagogy Faculty. Unfortunately, there was no time to study the pattern measurement unit (meter) from the need to communicate the result of the measurement, so that Teacher Mila could understand exactly what the measurement of the jump distance was, since there would be no way for her knowing the measure of the length of the string used by Liandra. Therefore, our suggestion for future experiences with the triggering situation in reference is to problematize the need for standardized measurement unit, its multiples, and submultiples. Discussion presents the research findings, as follows.

7.11 Discussion

Triggering Learning Situation entitled Liandra's request for help to students of the Pedagogy Faculty enabled students to carry out an objective operation: long jump body movement. This movement was important in interpretation process of the situation that gave rise the problem whose solution involves the number concept.

Length magnitude was the reference around which the reflections took place focusing on the relationship between the length of the jump distance (whole) and the measurement length unit (part).

Because such measures were unknown, they were represented in graphic and literal forms (Fig. 7.1). Relationship between the whole (jump distance length) and parts (measurement unit length) was revealed in the measurement operation (Fig. 7.2) and consolidated through the graphic model (Fig. 7.3).

Table 7.9 Answer letter to Liandra presented by group 1

Hello, Liandra. We are students of the Pedagogy Faculty, and we know the difficulties that remote classes resulted in So, we decided to help you with your Physical Education activity. We are aware of your inquiry regarding the jump you performed, in which you had doubts about the length measurement Firstly, so that you can measure the jump distance, you need to consider the magnitude (everything that can be measured or counted) involved, i.e., the length, a continuous quantity (it can be increased or decreased in small parts) therefore you must choose a standard measurement unit that will be used to measure the entire length We suggest that you use a malleable instrument (object), it can be a rope, a string, whatever you have at home that you can bend, which is flexible. It will be your basic measurement unit Then you need to know how many times the basic measurement unit fits within the total length of your jump, considering the rules of the long jump that must be measured from the starting point to where you marked and arrived Maybe the measurement of your string does not fit the length, and then you will have to subdivide it (dividing basic measurement unit again) You can use a letter to represent the basic measurement unit, the jump length, and the intermediate measurement unit (subdivisions) you use For instance, we assume you choose the letter M to represent the total jump length distance, the letter X to the basic measurement unit, and $\frac{1}{p}$ to represent subdivisions So, you need to measure how many times X fits into M, and if you need any subdivision, you will use the intermediate measure $\frac{1}{p}$ Finally, we have a universal model that you can apply in any length measurement situation: $M = X.\,N + \frac{1}{p}.\,Y$, where **M**: Total length distance; **X**: Basic measurement unit; **N**: Number of times the basic measurement unit is repeated; $\frac{1}{p}$: Intermediate measurement unit; **Y**: Number of times the intermediate measurement unit is repeated We hope we have helped you with your problem with measuring your jump From now on you can use this formula in any situation regardless the distance of any length to measure A big virtual kiss from the students A_{16}; A_{11}; A_5; A_{20} e A_4

Note From Rosa et al. (2021)

From this operation, a third element arose: the number of times the measurement unit fits into the magnitude measured. In other words, this is the result of mediation process. Measurement result appears as a number. Here, the number emerged as a singular form of general relationship representation between the magnitude measured and measurement unit. When varying the size of the measurement unit and/or the magnitude measured, singular result may also vary, as occurred in the fourth episode.

Graphic model (Fig. 7.3) is constituted as general base for four particular manifestations of the number concept expressed through multiplicity and divisibility relations (Fig. 7.4). Each element is a singular manifestation of a particular relationship. For instance, number t is a singular form of manifestation of particular relation of multiplicity between the values p and u ($t = p.u$).

Assuming new conditions experienced by Liandra allowed students deducing fresh ways of solving the problem from general relationship. In previous episode, numbers were revealed as singular manifestations of particular relationships of multiplicity and divisibility sustained in general relationship between whole and parts. Arrival point in previous episode is the starting point for the continuity of reflections. After revealing the numbers, students proceeded to operationalize them, in addition and subtraction context. Thereunto, they established the intrinsic relationship of the body movement performed during the jump with a model that fixes the whole-parts relationship in subtraction operation context (Fig. 7.6). Next, they recorded the general basis of the whole-parts relationship through arcs and segments (Fig. 7.7) and deduced from it four of its particular manifestations ($v = t - n$; $n = t - v$; $t = v + n$; and $t = n + v$). Similar movement happened in the second episode, but in multiplicity and divisibility relations context. Therefore, the cyclic character of the movement between general and particular was detected.

Adding new problem conditions enabled deducting new solution methods to the initial problem, including the synthesis of general solution procedure, valid for any measurement unit, even standardized ones.

In the fourth episode, with variation in the jump length measurement, the measurement result changed as well. Its singular expression was then in fractional form, but the general way to obtain the number remained: multiplicity relationship and divisibility between magnitudes.

In the fifth episode, based on the reflections carried out throughout the process, students prepared the answer letter to Liandra. Answer letter was an important evaluation instrument on the students' learning process on the general procedure for solving the triggering problem.

7.12 Conclusion

Development was carried out with future teachers exactly as it is done with children in Basic Education because we understand that rethinking the organization mode of teaching is necessary, both from the content viewpoint and the method.

Body movement of the jump constituted the concrete starting point. Reflections on body movement subsidized revealing the essential relation, its abstraction, and generalization through modeling process. As expressions of abstractions and generalizations means, the models were mediating elements between the concrete starting point (real concrete) and the concrete arrival point: answering the triggering problem based on the reflection synthesis carried out throughout the process.

Therefore, the conclusion is that the mathematical thinking movement covered by future teachers, when developing a situation that triggers learning, followed from the real concrete to the abstract, and from abstractions to concrete as a synthesis of multiple determinations. The teaching experience concretely lived based on length magnitude was paramount to guide reflections performed at abstract level.

Triggering Learning Situation is not limited to a situation that problematizes and reproduces human needs that historically gave rise to mathematical concepts, and it also aims at problematizing needs which afflict humanity in contemporary times. The plot of triggering situation Liandra's request for help to the students of the Pedagogy Faculty makes it possible to reflect on some educational inequalities in the pandemic context: children from a more privileged class had access to computers and applications that enabled remote classes, while others only received tasks via WhatsApp to develop with their family's help. Families are often illiterate, without time to teach their children at home, without food to supply the lunch offered at school, among other factors. Our intention, in future Triggering Learning Situations, is to make more explicit, not only historical genesis of mathematical concepts, but also the genesis of social inequalities. They must be at the base of the elaboration process of the problems that trigger learning.

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Chapter 8
Teaching Mathematical Problem Solving in Elementary School

Yolanda Rosas-Rivera, Yulia Solovieva, and Luis Quintanar Rojas

Abstract This chapter shows a qualitative analysis of teaching methods of problem solving in mathematics classes in primary school, specifically in the third grade of primary school. The theoretical framework of reference was the historical-cultural approach and activity theory. The participants were three teachers of the third grade of primary school and their corresponding pupils. The work consisted in (1) application of interviews to teachers, (2) observation of teachers and (3) evaluation of the pupils. The results show favorable and unfavorable conditions in the work with the solution of problems in classes. It is concluded that problem solving fulfills a developmental objective when the formation of mathematical concepts is considered. Provided analysis allows to arrive to three main conclusions about the relationship between the process of teaching and learning, organization of the solving of problems as intellectual activity and proposals for teaching of mathematics at primary school.

8.1 Introduction

The teaching and learning of mathematics is the common process of the work with knowledge, in which teachers and pupils take part. This process requires various activities and actions with objects that favor said construction. There are various factors that have been investigated, e.g., teaching practices, learning strategies, the use of teaching materials, the participation of non-formal knowledge, etc. Therefore, it is possible to have an idea of the complexity that implies for teachers and researchers to understand both the content of mathematics and the way of teaching it in the

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classroom, especially if it is in the early years, when the child's thinking maintains immediate characteristics of reality (concrete thought).

According to Cantoral et al. (2005) for there to be teaching and learning, it is necessary that knowledge be an important object of the interaction between teacher and pupils; an example of interaction proposed by the authors is the game. However, it is possible to identify that although there are changes in the teaching proposals, didactic materials and use of technology, as well as in the objective of the content of the teaching of mathematics (algorithms to mathematical concepts) (Ávila, 2006), the results obtained from these modifications are not yet perceived in the understanding of teachers (Árevalo, 2015), the learning of pupils (Rosas & Solovieva, 2019) or in their psychological development (Solovieva et al., 2021).

In this way, the objective of this chapter is to analyze teaching practices in primary school, specifically in the subject of mathematics. The first part shows educational research that is used as a theoretical and methodological basis. The second part describes the methodology, from a qualitative approach. Thirdly, the main findings are emphasized and finally the contributions and limitations of this study are analyzed and compared.

8.2 Part 1. Educational Research

The proposal for the analysis of teaching practices in the classroom has been worked on by CINVESTAV in 2017 (Weiss et al., 2019), considering ethnographic and didactics as the main analysis in subjects such as Mathematics, Spanish and Science in education. basic. The investigations proposed the use of classroom observation, interviews with teachers, pupils and school administration. The main findings in the mathematics classes in the specific work with the resolution of problems were: the majority of the problems had an isolated content and with the purpose of applying the knowledge taught (rather than the learned), problem solving in a group, in which the teacher made the main decisions and the pupils only gave their approval, however, few teachers worked with innovative problems and the creation of problems by the pupils, which according to Tsvetkova (1999) is one of the main learning activities.

Another study that proposes the same line and research tools is that of Reséndiz et al. (2017) to analyze didactic strategies in a multigrade group. The results of the analysis of the problem-solving tasks showed: (1) content of the problems: nature of the numbers, the number of operations, (2) types of help that the teacher gives: clarify the problem (what is asked, what known), check or recall the procedure, and check the result. It is possible to return to the challenges that arise in these classrooms, the conclusion of the study also rescues the teacher's training and her effort to generate learning activities through problem solving.

Problem solving implies a constant interaction between teachers and pupils, the authors Donoso et al. (2002) propose the following moments of analysis of mathematics classes in primary education: (1) beginning, implies all the exchanges prior to the problem-solving presentation (class objective, prior knowledge, methodology and inquiry), (2) development, interactions during the problem-solving presentation (problem presentation, problem analysis, problem execution, problem verification and argumentation, (3) final, what happens after solving problems (integration and assessment). Considering this structure, the authors analyze 23 audiovisual records of classes of fifth and eighth grade mathematics teachers from public schools in Chile. The results obtained were: (1) all the teachers identify the previous knowledge of the pupils, however 16 of them do not express the objectives or expected learning of the pupils. Other activities that were less worked on during the initiation phase were the precision of the work methodology and the inquiry (questions about the topic, the relationship of the topic with the resolution of problems), (2) in the development phase, the teachers present the problems, whether they read it, analyze it and execute it, the teacher does not inquire about the understanding of the problem before proposing the solutions, as well as fails to relate the answers of the pupils with the learning objective, inquire into the nature of the problem, in the selection of the solution plan and organizing the solution strategies, and (3) in the closing phase, the teachers integrate the contents and carry out the assessment (relation of the content with daily life and recognition of the work of the pupils).

The analysis of the previous research work describes the actions worked during the solution of mathematics problems in basic education, and it is possible to agree in identifying the little or no orientation and interaction between teachers and pupils, which does not favor their cognitive development.

A more detailed study on the orientation processes and that is related to the analysis of the structure of the problem-solving activity has been investigated by Nikola and Talizina (2017). The authors have proposed problem solving as an intellectual activity, which contains psychological actions that can be taught as general procedures. These actions are orientation, execution and control and will also be described in the following section on theoretical concepts. Now we are interested in showing that these general procedures have been investigated in the solution of arithmetic problems (Rosas, 2012; Rosas et al., 2017) and solution of geometric problems (Volodarskaya & Nikiluk, 2017). Both proposals consider the cultural historical approach, specifically the activity theory applied to teaching (Talizina, 2019).

The proposal of Nikola and Talizina (2017) consists of organizing the content and the actions necessary to solve arithmetic problems that include the processes of speed, distance and time. The authors use an experimental formative method with the following activities: (1) formation of generalized concepts about the three basic magnitudes (time, distance and speed) that constitute the specifics of all the situations described in the problems, (2) the assimilation of the relations that exist between the indicated magnitudes, (3) the assimilation of the relations between the particular and general meaning of each magnitude and (4) the formation of the general method for the moderation of any type of situation that is described in the problems.

The first step consisted of organizing the concepts of the processes (time, distance and speed) and they developed tasks to form these concepts in the pupils through the theory of the formation of mental actions (Galperin, 2016; Solovieva & Quintanar, 2019). For example, the concept of time was worked on from: identification of the start and end of time measurement, identification of time units (time measurement).

The authors proposed the use of special models, time was represented in a segment of the straight line, thus 1 h was represented as a 1 cm segment. After orienting the model, the authors asked them to represent 4 h, then the first hour, the second hour, the last hour, the first two hours, the beginning of the measurement, its end, etc. These tasks allowed the pupils to differentiate from a temporal interval, both the moment of time and the ordinal number of the time unit. Subsequently, once having worked on each Magnitude (time, speed and distance), the authors set out to organize and develop the general conditions to solve problems that these Magnitudes imply.

The main task of the general procedure for problem solving was to teach schoolchildren to build the problem situation scheme and the problem-solving scheme, draw up the plan and choose the means of solution. Nikola and Talizina (2017) explain that the elaboration of the schema for the situation represents the translation (comprehension) of the text of the problem in the language of the concrete abstract model. In this model, all the concepts and relationships that participate simultaneously might be identified. In addition, it is necessary to consider the rules that determine these relationships, for example, in the case of the concept of time, it is necessary to know if there are one or more than one processes of time duration in the problem, if they complement, diminish or act together. In conclusion, the generation of the scheme allows one to abstract from the specific data of the situation and see the given problem, as the single case of the relationships between the concepts addressed in the problem situation.

From this research, it is possible to identify those orientations and contents that make up the solution of mathematical problems. The main actions being: (1) analysis of the situation of the problem, through the identification of sufficient and necessary data, and relationship between the data, (2) elaboration of the solution scheme (relationship between concepts and their materialization), (3) elaboration of the plan solution plan and (4) execution of the solution plan. All these actions can be directed by the teacher.

In summary, it is possible to recover the solution of problems as an intellectual activity, although it seems obvious that this is the case, we can identify in various theoretical or practical proposals that study the solution of problems only such as finding the mathematical operation and giving algorithm solution answers (Rosas & Solovieva, 2019). It is for this reason that, in the following section, conceptual framework, the main characteristics of problem solving will be described.

8.3 Theory of Activity Applied to Teaching

The need to carry out an analysis from psychology has been identified by authors such as Talizina (1988, cited by García & Tintorer, 2016). The authors García and Tintorer (2016) summarize in three main points the criticism towards the positions of methods established by Poyla, from mathematics: (1) the solution of mathematical problems is based on operational rules without considering a learning theory, (2) it does not identifies the characteristic of transformation of the actions (material-mental), (3) they are not considered means of support for the learning management process, and (4) the approach of steps that they propose does not guarantee the efficiency of the process of acquisition of intellectual activity (Solovieva & Quintanar, 2021). Based on this analysis, this chapter will review the theory of development and learning established by Vygotsky (2006) and his followers Davidov (1988), Galperin (2016) and Talizina (2019).

In general, there are two perspectives on the path of assimilation of the content of mathematics, the spontaneous path, and the directed path. According to Talizina (2019), the spontaneous pathway is related to Piaget's theory of development (1975), who establishes that the child acquires knowledge from the biological basis, in this way it is not necessary to organize the teaching of knowledge because the child when he is prepared will be able to assimilate the mathematical concepts (Solovieva et al., 2022). However, from his epistemological perspective, Piaget considers that mathematics is assimilated by children from logical operations that they acquire through the use of concrete objects. These logical operations consist of seriation, conservation, classification and the synthesis of said operations correspond to the concept of number. In addition, Piaget (1975) considers that when solving mathematical problems or tasks it is necessary to understand the child's solution process, for which reason the adult should always establish questions that allow the child to express his or her arguments (Kamii, 2003; Kamii & Dominick, 2010). In conclusion, Piaget's mathematics teaching proposal consists of discovering the knowledge and strategies that each child creates to solve mathematical problems or tasks, it is important not to limit their thoughts to the adult experience. Mathematics is a type of abstract knowledge that requires mental actions carried out by the child himself.

Continuing with the second way of assimilation (directed), this perspective is based on the following thesis established by Vygotsky: psychological development from the beginning is mediated by education and teaching (Davídov, 1988). Vygotsky (2006) specifies in this regard that teaching should not be directed toward the past but toward the future of the child's development, creating the zone of proximal development. The objective of the teaching–learning process is to organize the necessary conditions (pedagogical, psychological) to achieve psychological development of the pupils as qualitative changes in the cognitive sphere but also in the affective-emotional sphere. These changes will not appear spontaneously because of maturation, hence the importance of knowing the postulates of this theory (Talizina et al., 2010).

In order to guarantee the pupil's acquisition of mathematical concepts and actions, the teacher has to provide precise knowledge on the content, structure and necessary characteristics of the actions that are intended to be developed in the pupils, as well as organizing the academic content in stages: material, perceptive, external language and internal language (Galperin, 2016; Nikola & Talizina, 2017; Rosas et al., 2017; Talizina, 2019).

According to Talizina (2019), during the process of organization of teaching, it is essential to train specific skills of cognitive activity that go beyond the limits of the academic subject, but at the same time, determine the success of their mastery. Such is the case of problem solving in school sessions at the primary level. This cognitive activity requires not only knowledge of arithmetic, but also an understanding of the essence of the basic elements of the situation in question and the relationships between them (Vicente et al., 2008).

In traditional teaching, the solution process is normally not disclosed to the pupil, so awareness and reflection do not arise. The pupils act in a chaotic, empirical and not very reflective way. Sometimes, they just may recognize the failure of this difficulty. The solving of problems is not even considered as an activity in school sessions.

From the point of view of the reflexive organization of the activity (Nicola & Talizina, 2017), during the process of solving pf problems, the pupil must start by determining what actions he is going to carry out. To do this requires a reflective identification of the problem question and the conditions under which the question is established. The conditions of the problem always describe one or another situation, behind which the pupil must discover the determined arithmetic relations, that is, he must describe the situation that is mentioned in the language of mathematics. The teacher should not simply present and announce the problem, but also show what needs to be done for the pupil to solve any problem. For this, it is necessary to know what type of intellectual actions the process of solving any problem of a given school grade consists of and in what order they must be carried out (Talizina, 2019).

In relation to problem solving, it is emphasized that each problem includes a fundamental question, the answer to which constitutes the objective of the solution. In order to answer the final question (or intermediate questions) of the problem, certain data must be analyzed, that is, obtain the necessary and sufficient information from the mathematical and logical links formulated by the problem. This is obtained through the orientation-investigative activity, which constitutes one of the components of intellectual activity (Galperin, 2016).

Only on the basis of the data, obtained from the orientation-investigative activity, the pupil can elaborate the general scheme for the content of the problem, establishing the data, the essential relationships between them and the possibilities to answer the final question. It is possible to find the arithmetic operations that correspond to the created plan (Tsvetkova, 1999). The teacher should not simply present and announce the problem, but also show what needs to be done for the pupil to solve mathematical problem. To do this, it is necessary to know what type of intellectual actions are included in the process of solving any problem of a given school grade and in what order they must be carried out (Talizina, 2019).

8.4 Analysis of Teaching Methods of Problem Solving

This study will be presented in a more expanded way to contribute to qualitative research on the teaching–learning process of mathematics. Selection of educational institutions was based on the proposals and on voluntary disposition of the institutions to take part in this intentional empirical research. Three private sector schools in Mexico City and in the city of Puebla were chosen, each school used particular method for teaching mathematics.

These teaching methods used by schools are based on the plans and programs established by the Ministry of Public Education of Mexico (SEP, 2017). However, the private schools may organize teaching practices and teaching activities considering the pedagogical approaches related to the philosophy of each school. In other words, the schools share a graduation profile for primary education, which is established by the SEP (2017), but each school has the possibility of organizing its own forms of teaching.

The first private school, located in Mexico City, uses the concept of competencies as a pedagogical approach, carries out the certifications and courses taught by the SEP (2017). The director and the teacher called this approach as "official" because all teachers have certifications and take part in all activities proposed by the SEP.

Additionally, this school uses only those pedagogical resources, proposed by SEP (2017). This school is called as Official school for convenience of the study.

The second private school, located in the city of Puebla, uses Montessori approach. This school has certifications and professionalization courses on philosophy and Montessori Teaching. In addition, it has contacts with National and International Institutions, which use only Montessori pedagogical perspective for teaching process. This school is called as Montessori school for convenience of the study.

The third private school, located in the city of Puebla, is unique school in the country, which uses cultural historical approach and activity theory as the basis of general teaching method. All teachers take part in professionalization courses and learn Vygotsky's pedagogical perspective. The school administration, the teachers and advisers take part in national and international events and research related to this approach (Solovieva et al., 2017, 2021). This school is called as Vygotsky's school for convenience of the study.

It is possible to identify that the Montessori and Vygotsky schools handle very specific methods related to pedagogical perspectives and psychological development. In addition, both schools propose specific methods of teaching mathematics in basic education.

In all selected schools, the teachers reported satisfactory results on the pupils' learning of mathematics, in addition to expressing their interest of participation in research activities. Finally, a voluntary participation agreement was made and the main benefit for the schools was to know those teaching factors that did favor the learning of mathematics in the pupils, and those strategies that could be improved. Table 8.1 presents the characteristics of the schools and participants.

The methods of the research were the interviews with the teachers, non-participant observation and qualitative assessment of the pupils.

8.4.1 The Interviews

In each school, the interviews with the teachers were carried out. The topics of interviews allowed to identify the teachers' knowledge and skills to plan and carry out problem solving.

Table 8.1 Characteristics of the schools

Schools	Location	Number of pupils	Average age of the pupils	Teaching method
Official	Mexico City	8	8.5 years old	Based on the official program (SEP-competitions)
Montessori	City of Puebla	5	8.5 years old	Based on the Bancubi material and Montessori method
Vygotsky	City of Puebla	6	8.7 years old	Based on cultural historical approach and Vygotsky's proposals

Table 8.2 Characteristics of the teachers

Schools	Official (S)	Montessori (M)	Vygotski (V)
Age	43 years old	38 years old	41 years old
Academic degree	Degree in education	Degree in Pedagogy	Degree in Education
Teaching group	Third and fourth grade	1st to 6th grade	Third to fifth grade
Teaching experience	6 years	5 years	13 years
Training courses	"CONOCER" (SEP)	Bancubi Method	Courses: development of number concept, drawing activity and story

The three teachers reported participation in constant training for teaching mathematics from the perspective that they work in the corresponding schools and have spent an average of 5.6 years teaching from the school's pedagogical proposal. All three teachers reported having studies in education and experience as primary school teachers in private schools. Even though the teacher from Vygotsky school has 13 years of experience, the teacher acknowledges to study and use this approach for 6 years and since then she has considered this theory to organize her teaching tasks.

Table 8.2 describes the characteristics of the teachers.

The interviews lasted an hour and a half. The main contents addressed were as follows: conception of mathematics, systematization of content, types of orientations, types of actions, mathematical concepts, and skills (problem solving) and updating activities.

8.4.2 Non-participant Observation

This activity allowed us to relate the planned activities and those that are carried out in the classroom. A log was used with the categories: content of mathematics, systematization of content, types of orientations and actions. It was agreed to observe three mathematics classes, in which problem solving was addressed. The duration of each observation was approximately one hour and a half. It was agreed that the researcher occupy a place that would allow the observation in a broader way.

8.4.3 Qualitative Assessment of the Pupils

Qualitative evaluation from the approach of Vygotsky (2006) and Talizina (2000) allows us to understand the process and dynamics of solving mathematical problems. Talizina (2000) refers to the following characteristics of the evaluation process: (1) the evaluation of operations, objectives and reasons for problem-solving tasks is carried out, (2) identifying the actions of the contents (identification of the final question-orientation, identification of quantitative relationships-execution and answer to the final question-verification), (3) the tasks must vary according to the content to identify the degree of generalization, (4) the tasks can be solved on various levels of execution (material, perceptual or verbal), (5) identify the degree of reflection in the responses.

The objective of qualitative assessment of the pupils consisted in analyses of the skills, strategies and types of difficulties and mistakes. Such evaluation provided the information about the contents and skills learned as well as the types of help they required to achieve the solution of mathematical problems, that is, we were able to identify the zone of real development and the zone of proximal development.

The evaluation lasted 40 min for the problem-solving section, it was carried out individually. In the evaluation protocol, a total of 10 mathematical problems were used. The types of mathematical problems are simple (one mathematical operation for its solution), complex (more than one mathematical operation for its solution) and no solution. In addition, they were divided into solution tasks: (a) problems given by the researcher, (b) creation of problems from data provided by the researcher and (c) creation of problems by the pupil himself or herself.

8.5 Results

The results are presented in two aspects, first describing the teaching process and then its relationship with the learning obtained by the pupils. It is important to stress that it was a big challenge to argue that the learning process depends predominately on the teaching method used by the teachers. This analysis includes the aspects of psychological development of the process of problem-solving, the planning of intellectual actions and acquisition of concepts by pupils.

8.5.1 Problem-Solving Teaching Process

Analysis of semi-structured interviews and observation of classes allowed to characterize the teaching proposal of the teachers of the three schools. Table 8.3 gives the general elements that teachers consider for teaching mathematical problem solving.

The teachers were able to express their main objective of teaching mathematics. The teacher of the official school (S) said that her main interest is that the pupils "solve the operations of addition, subtraction, multiplication and division with numbers and fractions" and the way to know if they achieve that objective is by solving tasks and math problems. She also commented that the strategies that she uses for children to solve problems consist of asking the pupils to read well, to underline key words such as "more" "she divided" "decreased" that are related to arithmetic operations. In addition, in mathematics classes, it was possible to identify her work dynamics. The teacher gives them the following problem: "Ramiro has 360 packets of peanuts in a sack, if he is going to distribute them among his 12 relatives, how many packets will each one gets? Do you have too many? How many?".

The teacher solves it with her pupils, she writes on the blackboard and passes the pupils to solve the operation. The teacher writes on the blackboard: data, operation and results. She then interacts with the pupils in the following way:

- **Teacher**: tell me (looking at the pupils) what are the data? What is the problem talking about? The numbers what?
- **A pupil**: "packages of peanuts"
- **Teacher**: packets of peanuts, how many? Tell me the number, and then, among twelve, I'm going to put it here, I hope you understand it, then we're going to see the operation, of course it's…
- **A pupil**: 30
- **Teacher**: no, the operation
- **Another pupil**: 360 divided by 12,
- **Teacher**: Exactly, how did you do this? Right now, if I didn't ask you, it's as you would like with subtraction or without subtraction (division algorithm by subtraction or use of multiplication).

Subsequently, the operation (multiplication) is verified, and the problem is considered solved. Figure 8.1 shows the execution of a pupil from this school. The dynamics

Table 8.3 Elements of teaching the solution of mathematical problems

	School S	School V	School M
Mathematical content	• Empirical (operative) • Operational skills for problem solving	• Scientists (number) • System of concepts (number, number system, fraction) • General problem-solving skills	• Empirical (operative) • Definitions: covering isolated characteristics

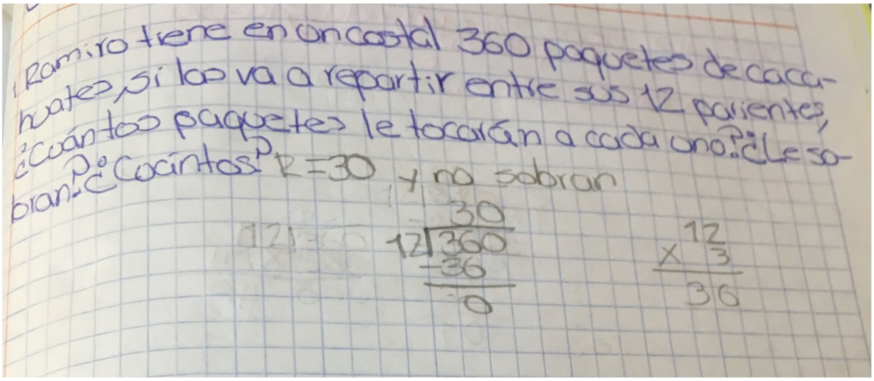

Fig. 8.1 Example of the solving of mathematical problem

for other mathematical problems are the same: the teacher asks the pupil to read the problem, asks questions about the data and writes them on the blackboard, asks the pupil to go on to solve the mathematical operation and ends with writing the result.

In our analysis, this dynamic only corresponds to the work of mathematical problems from actions with algorithms. The teacher guides the pupils towards the identification of quantitative data, mathematical operation and result of the mathematical operation. The pupils only answer quantitative information, a reflection on the whole situation of the problem and how the data is related to propose a solution is not achieved. In addition, the verification of the solution of the problem is based on solving another algorithm, but it does not analyze why it should be done. It was also possible to identify that the only interaction proposed is teacher-group. Finally, it was also possible to check that the algorithms that the pupils copied from the blackboard had some errors related to the positioning of the digit.

The next school that we present is the Vygotsky (V) school. The teacher shared her teaching objective. She mentioned that her main interest is "the development of mathematical and conceptual thinking, e.g., that primary school children form the concept of number… problem solving is aimed at reasoning and involves giving an answer to a question, an analysis of data, likewise not all problems have a solution, there is data that is left over, that is tricky".

The teacher reports that she uses problems (written or dictated) with everyday situations, sometimes the pupils create the problems. During class observation, it was possible to identify the following dynamics: the teacher dictates and writes the problem on the blackboard, she waits for the pupils to write. The problem was the following: "Isa, she has a box with 2556 apples, five ninths are red, and the rest are green, how many red apples are there? How many green apples are there? How many apples are there?".

The teacher reminds them that they are going to work with the orientation card. The teacher distributes the orientation card to each pupil, so that they have it on their desk during the work. Subsequently, the teacher asks, "someone help us read the problem-solving card". One pupil participates and reads the content of orientation card loudly: "(a) read the problem, (b) with your words mention what is being asked in the problem, (c) write what data you know, (d) answer if with the data you have, you can answer the question, (e) choose the necessary mathematical operation and write the formula, (f) carry out the necessary steps to solve the operation, (g) check the result, this result should answer the final question of the problem".

Further on, the teacher emphasizes these steps, explaining each one to the pupils: "we are going to work in teams to be able to solve, when they have the work we will verify it on the blackboard… okay… we are doing it… data, formula, operation and result, and then what we work on is our schema and we represent it: (a) first we have to read our problem, (b) mention what is missing to answer the question of the problem, (c) write what data we know, remember that there are problems where there is excess data, where data is missing, and where what is needed is, remember that there are problems that cannot be solved (data is missing), (d) then we have to respond with the data, of the ones we have, can we answer the question?, (e) after choosing which operation we are going to use to solve the problem, and (f, g) well… and after following the necessary steps, we see our unknown, and with our operation we give the result to the question of the problem".

It is possible to identify that throughout the procedure and the interaction with the pupils, the teacher shows a problem-solving structure, which begins with the identification of the final question of the problem. The organization of data, operation and results is also worked on, but the teacher uses constant reflection to identify relevant data and their relationship, not only mathematically but also logically, when she comments that the characteristics of the data (necessary data, missing, insufficient). During the work, the teacher uses the actions in the materialized plane. The teacher gives the pupils the pieces of leaves to represent the measurements of fractions. The pupils had to represent it through their scheme and solve it in an external verbal plane (writing). It is also possible to observe that the teacher uses the pupil-object interaction when she asks them to materialize the measure; the teacher also used pupil-pupil interaction, so that the problems were solved in pairs and then in groups (Fig. 8.2).

To conclude this section of analysis, the M school is described. The teacher of this school refers that the objective of mathematics is "the development of quantitative and logical thinking skills, e.g., reasoning, probability, statistics and logic" in addition, conceptualizes the number as "a friend/enemy, they are part of daily life, e.g., the monetary weight". To achieve this teaching objective, the teacher considers: the attitude, the way of presenting the material and its use, the accompaniment and correction, as well as the constant observation of the pupil's disposition. Finally, the teacher herself refers that teaching "is very sensory."

In this school, no specific way of working with problem solving was identified. The teacher expressed that this activity was carried out only spontaneously with everyday situations.

For example, "if we are in the dining room and I arrive with cookies I tell the pupils, I have 6 cookies and only three people, how many cookies will each of us have to eat?" So "children do not use algorithms, but what they observe and distribute the

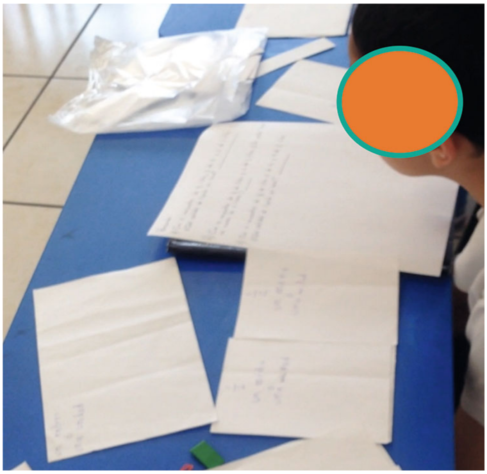

Fig. 8.2 Use of materialized actions in solving of the problem

cookies can be one by one or some already do two by two." For the teacher, the main objective is to know the answers of the pupils, how they build their results from their experiences and knowledge obtained. In addition, the teacher commented that the teaching of mathematical algorithms is taught up to sixth grade, so solving problems with mathematical operations is worked on up to that grade. The mathematical tasks that are worked on in the classroom is based on the exploration of the Montessori material.

During math classes, it was possible to observe materialized actions. The pupils always begin their learning with the use of the Bancubi material. The tasks they perform with this material are as follows: to form quantities, to compare quantities, operations of addition, multiplication, distribution, and some geometry tasks (calculation of area and volume). The pupils show pleasure toward the use of the material, some play with the material before listening to the teacher's instructions, but after the repetition of the indication, they do what is requested. In addition, they asked to use their material during the dynamic evaluation.

Sometimes the work in the environment is done individually with their Bancubi book, the pupils sit where they like the most and each one works on the subject they are learning. One pupil was working on equivalent fractions tasks, in the Bancubi book, he was told to create a model with his cubes, similar to the exercise worked on with his teacher. The pupil formed these models with his cubes, then he had to divide the whole into as many parts as requested and color the corresponding part in his book. Likewise, in this book, questions are raised that relate the parts to the whole, in which the pupils not only answers with the quantities but writes their solution process. The pupils usually comment aloud on the actions they are carrying out, they also tend to explain some actions to their classmates and ask the teacher specific questions.

Figure 8.3 shows an example of a task from the Montessori book.

Make with your cubes	The multiplication is…	First factor	Second factor	Product
2424 x 2424				

Fig. 8.3 Example of the task from Montessori book

During the analysis, it was possible to identify the three proposals on the teaching of the process of the solving of mathematical problems:

(1) A teaching method is oriented toward the operational part, that is, it focuses on the solution of mathematical operations. The solving of problems aims to find the quantitative data and relate them to the arithmetic operations, and the problem is answered when the solution of the arithmetic operation is found. In addition, this method makes use of figures and the solution of mathematical algorithms, without understanding the decimal number system or the situation of the problem (College S).
(2) A second method is oriented toward the development of mathematical concepts and focuses on the understanding and solution of the problem question. It makes use of the components of quantity, measure and magnitude. Mathematical actions are results of the relationships of said components (College V).
(3) The third method consists of the development of the pupils' own strategies, the use of their knowledge to solve everyday situations. The teacher raises everyday situations verbally and each pupil looks for a solution, also verbally (College M).

The following section will describe the results of the dynamic evaluation applied to the pupils of the three schools.

8.5.2 Dynamic Evaluation of the Process of Problem Solving

This section consisted of reading simple or complex mathematical problems to the pupil and asking them to solve. If they needed to write, draw or ask for a word they did not understand, they could do so. In general, the pupils liked this task, most of the pupils asked for the problem to be repeated and they did an active repetition of it, when the problem was simple, they repeated the data but when it was complex, they repeated most of the tasks the problem situation. The task that the pupils from the schools M and V liked the most was writing problems for their classmates, while in school S, the pupils had strong difficulties collaborating on these tasks, they sought to do complex problems for their classmates, showing some competition between them.

Table 8.4 gives the types of errors and the external orientation needed by the pupils to solve the problems. In total, the pupils worked with 11 problems. The types of problems posed included the four mathematical operations and can be organized as: (1) simple, involving a mathematical operation, with a variant in everyday situation (books/shelves) and non-everyday situation (km/h), example: "A train it advanced 98 km during 8 h. How many kilometers does the train travel in an hour?" (2) complex problems, which imply the solution of two operations (measurement conversion, inferring data), e.g.: "Renata and Daniel went to the market and bought the following: 2 kilos of apple, 300 g of sugar and 1 kilo of pasta, how many grams did they buy in total?", (3) problems without a solution, which had a situation and a final question plus these were unrelated, e.g.: "During 12 days, it was built a highway of 48 km, how many cars passed during a day?, (4) creation of problems from data and free creation of problems".

The results of the pupils from school S and school M had difficulties in the actions related to the understanding of the problem (orientation), specifically in the identification of the initial question and identification of relevant information. Due to this, the pupils form these schools had difficulties to recognize the problems that did not have a solution, they only gave an answer to the mathematical operation, but they did not understand that the situation of the problem was not related to the question of the problem. These pupils have also committed mistakes during elaboration of the solution plan and execution of the plan (Table 8.5).

The pupils of school S responded verbally to the mathematical operations of the problems based on the estimation of results and the use of counting their fingers. When they were asked to write the mathematical operations, the pupils had significant difficulties, which consisted in the absence of any knowledge of the solution of the algorithm (multiplication and division), organization of the algorithms, number hierarchy, logical relationship between the data. The pupils of school M did not use algorithms, since they are taught until sixth grade, but they asked to use Bancubi cubes and performed addition and subtraction operations.

However, the errors were also identified when handling the measurement, and the pupils had confusion between the colors of their cubes and their measurement value, as well as in the conversion actions of the measurements, e.g., when they had to convert hundreds to tens or tens to units. The count was also carried out by estimates but with the help of their cubes. In relation to the verification element (control), the pupils from both schools (S and M) had difficulties in answering the question of the problem, they only mentioned the result of the mathematical operation. In addition, the pupils from the school M require greater motivation to start solving problems, they were happy to attend to work with us but when the problems were read to them, they first waited for us to give them the answers, after asking them about the problem and how they would propose To solve them, the pupils began to repeat the problem as completely as possible and then used their material to carry out the operations.

Table 8.4 Results of the pupils during solving of simple and complex problems

Type of school		Official school	Vygotsky school	Montessori school
Difficulties	Aspects which required orientation	• Difficulty to identify the problem final question • Difficulty to identify the data and organize it (conversion) • Impulsiveness and doughs during	No difficulties and no orientation needed	Difficulty to identify the problem final question • Difficulty to identify the data and organize it (conversion)
Execution of plan	Executions of actions	• Estimate of results • Organize data in the algorithm • Hierarchical value of the number	• Identification of the measure in the data	• Use of addition or subtraction operations • Use of didactic material to solve the operations • Estimation of results
Verification	Control	• Answer the problem question • Verification of algorithm and problem results	No difficulties and no orientation needed	• Answer the problem question • Verification of algorithm and problem results—Answer the problem question • Verification of algorithm and problem results • Search for constant approval from the evaluator
Types of support provided by the evaluator		• Explanation of the use of the measure • Writing the conversion and organization of data • Repetition of the problem • Mathematical logic explanation	• Writing the conversion and organization of data	• Explanation of the use of the measure • Use of bancubi material • Concrete drawing support • Constant animation, support to continue with the task

Table 8.5 Responses of the pupils of three schools

Type of problem	Answers	
	Official	Montessori
During 12 days, a 48 km road was built, how many cars passed during a day?	(a) 48/12 (b) 48/1 (c) 12 + 48	(a) 5 (b) 12 (c) Depends on how much cement they are putting
Gaby is three times as old as her sister Sofía. If Sofía is 7 years old, how old is Gaby?	(a) 10 (b) 28 (c) 16 (d) 20	(a) 10 (b) 28
A train travels 98 km in 8 h. How many kilometers does the train travel in an hour?	(a) 48 (b) 11	(a) 11 and more 2 (b) 2 km (c) 1 km

The types of support for both schools consisted mainly in the organization of the problem understanding element (orientation), it was explained to them that mathematical problems consisted of two parts: problem situation and problem question. They were also given the logical and mathematical explanation of the problem information, always keeping in mind the problem question. The explanation of the mathematical operations was carried out with the perceptual support (concrete drawing) for the pupils from the S school and with the support of the Bancubi material for the pupils from the M school.

On the other hand, the pupils from the school V managed to solve all the problems, only one pupil had difficulties executing the solution plan, specifically the identification of the measure in the data. The problem involved several data with different measurements and the pupil operated with all of them without converting the measurements, so he was asked to write the data, and this helped the minor perform the conversion and solve it.

According to Tsvetkova (1999) and Nikola and Talizina (2017), the mastery of the use of mathematical concepts and skills consists of elaboration of problems with the data given by the researcher and the possibility of independent elaboration of the problems. Table 8.5 gives examples of elaboration of problems based on the data given by the researcher, while Table 8.6 presents the examples of the own elaboration of problems by the pupils are shown. According to the tables, it is possible to identify that the pupils from the schools M and S had difficulties in identifying the final question, as well as the relationship between the situation of the problem and the question of the problem. The pupils also had difficulty identifying the decimal number system.

The analysis of the data allowed to identify those elements of problem solving that are worked on in the classroom in each school. Firstly, it is possible to identify the knowledge that teachers have of mathematics and their main objective. The teachers

Table 8.6 Independent elaboration of the problems by the pupils

Collage	Data given by the researcher	Own elaboration
	16 units, 6 tens and 4 hundreds	
S	"Laura did an operation that has 18 units 6 tens and 4 hundreds, what is that number?"	"Juan has \$440 and he wants to buy marbles that cost \$20. How much money does he have left over?"
M	"If I have 18 ice creams and they take away 6 ice creams and give me 4 ice creams"	"If my dad has 172 candies and my mom gives him 129 and I give him 111"
V	"Isa has 6 tens of cards, 4 hundreds of cards and 18 units of cards. How many cards are there in all?"	"Juan fell asleep at 9:45 pm. And he woke up at 6:50 am, how many hours did Juan sleep?"

from the schools S and M do not consider the problem-solving activity as a teaching–learning activity, only as a "creative" activity (school M) or as an evaluation activity (school S). However, the teacher from the school V considered the process of problem solving as an activity with the structure and specific actions, which were observed in her classes.

Secondly, it is possible to identify the emphasis on the execution part of the mathematical problems. In the school S, the pupils were more concerned with solving the operation than with understanding the problem, and it was also observed that the pupils write their own problems they first chose the operation and then the situation. In contrast, the pupils form the school V achieved a better understanding and corroboration of mathematical problems, and their writing of their own mathematical problems was planned based on whether it would have a solution and based on the final question. The pupils from the school M have only managed to establish exercises, for example, how much is $400 + 27$? The third element that we can consider is the cognitive interest that arises when working with the solution of problems. The pupils from the school S were only interested in responding quickly and being the first to solve them, they were in competition mode with their peers. The pupils from the school M required more support to think of other ways to solve, they only proposed the use of their material and when they failed, they stopped admitting it. The pupils from the school V were interested in solving complex problems, they also focused on putting "traps" to the problems they elaborated.

8.6 Conclusions

Provided analysis allows to arrive to three main conclusions about the relationship between the process of teaching and learning, organization of the solving of problems as intellectual activity and proposals for teaching of mathematics at primary school.

1. The relationship between teaching and learning

Provided analysis allowed to identify the elements of teaching that are assimilated by the pupils. The proposals of the teaching methods of the teachers are based on particular perspectives of teaching. The teacher who uses the official competency model as a frame of reference only focuses on reviewing results without analyzing the cognitive or affective processes involved in solving problems. The teacher who uses the Montessori reference framework, has as its main objective that the pupils relate mathematical content with characteristics of the materials and associate it with deaths, so that the pupils have their materials as their main resource. Finally, the teacher from the school V, who uses Vygotsky as the main frame of reference, builds the main tools in the classroom, such as the problem-solving card and the proposed sheets that represent the measurements. What these three perspectives invite us is to question the content we are teaching and how we are teaching it. The children of these schools have a learning, the types of support were the difference between the results that they obtained. That is, the evaluator managed to make everyone understand the problem and its solution. To achieve this, the evaluator has mathematical, pedagogical and psychological knowledge, she managed the use of Bancubi materials and the use of algorithms. So, it is necessary to focus on where we want to take the pupil, to the present or to the future.

2. Problem-solving activity as an intellectual activity

The study shows that the best assessment task is to ask children to design their own problems, since this task requires specific knowledge of mathematics and their cognitive interests.

The problem-solving process has specific psychological structure that derives general skills. These allows the pupil to analyze and understand any problem, identify relevant information and solve it reflectively. Knowing this structure allows teachers to have a greater chance of developing pupils' thinking. The process of formation of mathematical concepts is required during solution of mathematical problems. The use of everyday situations is not entirely favorable for the development of scientific concepts, because no kind of generalization of the knowledge is used by teachers in primary schools. On this topic, Valdés (2017) has mentioned that teaching process should choose between presentation of solving as theoretical process or as practical examples of concrete solutions and that it is extremely difficult to find teachers who use both ways of teaching.

3. Proposals for teaching of mathematics at primary school

It is important to consider the present study might help to elaborate proposals for the analysis of mathematics teaching methods in primary school. The use of dynamic and qualitative research methods allows to provide further analyze the interactions and contents that arise between teachers and pupils. The collaboration between teachers and pupils is essential during teaching and learning process, which might not consist only in common interact action, but also in joint and orientated work during the process of solving of mathematical problems. The process of teaching and learning must not be considered as two isolated activities, but as a unique form of collective activity (Morales González et al., 2021).

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Part II
Research and Proposals for Teaching of Mathematics

Chapter 9
Development of Mathematical Skills of Primary School Children with Different Levels of Executive Functions in CHAT-Based and Traditional Programs

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Abstract This chapter presents the results of a longitudinal study that examined the development of mathematical skills of primary school children in relation to two types of factors: the child's educational environment and individual cognitive characteristics. In terms of the educational environment, the participants varied in the curriculum they received—a program built according to the principles of Cultural-Historical Activity Approach (CHAT) was compared to a traditional program. In terms of individual characteristics, children' executive functions were examined as a predictor of mathematical growth. The development of basic mathematical skills expected upon completion of first grade (7–8 years old children) was assessed: writing numbers in Base-10 number system, addition and subtraction (single-digit and double-digit numbers), comparing numbers in magnitude, measurment (to measure objects with a given unit of measurement), and finding the place of a number on a number line. This study revealed that the higher the level of development was, the faster the development of the two mathematical skills (writing numbers in Base-10 number system and measurement) developed. However, no significant dynamics in any other skills depending on the executive functions were registered. The CHAT-based educational program shows only a tendency to produce differences in the development of writing numbers in Base-10 number system, addition and subtraction of two-digit numbers, and finding the place of a number on a number line. At the same time, children with different levels of executive functions master mathematical skills equally within the two educational programs examined. Thus, it cannot be suggested that one program

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is more suitable than the other for children with a certain level of executive functions; children with different levels of executive functions develop mathematical skills at their own pace independently of the program they follow. The results of the study are analyzed in two theoretical perspectives of learning and development. The first one, Piaget's approach, emphasizes the need to adapt programs to the child's level of development. The other one, Vygotsky's approach, reflects the idea that training programs should be built in such a way that learning leads to development, and that those mathematical abilities that otherwise would not have arisen, are formed.

In recent years, the question of conditions and factors related to mathematical academic performance in the primary school has been actively discussed both in science and in educational practice. One type of relevant factor includes individual psychological characteristics, such as intelligence (Tikhomirova et al., 2019), self-regulation (Elhusseini et al., 2022), metacognition (García et al., 2016), motivational and personal characteristics (psychological well-being, motivation, and self-esteem indicators) (Gordeeva et al., 2018), and foundational mathematical skills (Krutetskiy, 1998; Nunes et al., 2012). The second type of factor includes contextual characteristics of the educational environment. These include methods, forms, and content of educational programs in mathematics, the specifics of performance assessment, and the time when a child begins to study mathematics (Watts et al., 2018). Personal and professional qualities of the teacher and the school educational environment are also taken into account. Finding an answer to the question of factors and conditions is of high practical importance, because it can help to determine ways of promoting mathematical development in primary school children. Nevertheless, the majority of existing research focuses separately either on individual or on contextual factors affecting the development of mathematical skills at this age, and the methodology applied is usually cross-sectional. In our perspective, it is important to investigate mathematical development longitudinally, in relation to both contextual and individual factors. Such investigation should be based on empirical findings from previous research, as well as a clear theoretical perspective. Multiple research studies have demonstrated an important role of children's executive functions in the acquisition of basic mathematical skills (Lucangeli & Cornoldi, 1997; Kyttala et al., 2003; Bull et al., 2008). With respect to contextual effects, educational programs can be expected to play a role in children's mathematical performance, although this factor has been relatively understudied. The conceptual system and the type of problems to be mastered by the child is of a special significance here. In this regard, the CHAT-based programs (Cultural-Historical and Activity Theory) provide a valuable comparison standard, since they imply a deeper understanding of mathematical concepts (Davydov, 1966; Aleksandrova, 2011). In the current study, we compare the mathematical growth in students receiving the CHAT-based instruction to that of students receiving a traditional instruction focused on the formation of specific mathematical skills.

9.1 Mathematical Skills in Primary School

Educational standards of different countries often include the list of compulsory mathematical skills to be developed by the end of the primary school (Primary School Curricula on Reading and Mathematics in Developing Countries, 2012; Mathematics Syllabus, 2013). The following requirements are common across many countries:

1. Rational numbers (reading, writing, comparison, regulation of numbers in the decimal notation, number line, grouping and classification of numbers based on different criteria) and operations (addition, subtraction, multiplication, and division);
2. Magnitudes, such as mass, length, perimeter, square, speed, time (reading, writing down, comparison), and their measurements by means of corresponding units;
3. Basic knowledge of geometry (recognition of geometrical shapes and determining size, calculation of length, perimeter, and area; evaluation of the figures' dimensions and distances between them);
4. Solving textual problems (establishing the relationship between values in the problem, planning the process of problem-solving, and arithmetical solving).

For the goals of the study, we made a list of skills that actively develop in the first grade and we followed the growth of these skills from the beginning of the school year until the end. Below is the list of the competences and skills we selected:

A. Symbolic numbers and their magnitudes

- Writing multi-digit numbers in the decimal system to represent a given number of objects, and the reverse skill: selecting the correct number of objects corresponding to a given multi-digit number (hereinafter, writing numbers in Base-10 number system)
- Estimating number magnitude by placing a number on a number line (hereinafter, working with a number line)

B. Arithmetic operations

- addition and subtraction of single-digit and double-digit numbers
- finding a missing part for addition and subtraction problems (e.g., missing addend)

C. Geometric reasoning

- Measuring the perimeter of a square using standard or non-standard units of measurement (hereinafter, the measurement)

9.2 The Relationship of Educational Program and the Development of Mathematical Skills in Primary School

Contemporary studies seldom examine the contribution of educational programs (considering together the contents, methods, and technologies) to the mastery of mathematical skills. Typically, separate characteristics of such programs are explored (e.g., see Hattie, 2008). Yet, it is instructive to study the influence of an educational program as a whole system on the development of mathematical knowledge. Therefore, it seems important to select programs that are conceptually different (i.e., have different content). Such programs can be found in the context of the Russian elementary school system. Several studies compared the academic outcomes of the CHAT-based programs developed by D. B. Elkonin and V. V. Davydov to the traditional ones (Tsukerman & Ermakova, 2003; Repkina, 1997; Pavlova, 2008, etc.). However, these studies had certain limitations. In particular, as pointed out by Gordeeva (2020), they utilized a cross-sectional rather than longitudinal methodology. Moreover, no assessments took place before the first school year, and many of these studies were conducted in laboratory schools, not the regular public ones. Besides, in these research projects, children were often asked to solve the problems that were typical for one of the educational systems under study (Sidneva et al., 2022). These features limit the ability to determine the extent to which any observed variability among students reflects a true difference in children's mathematical thinking. In the present study, the previous limitations were addressed by using a diagnostic toolkit, developed independently of specific educational programs, to longitudinally examine the growth of mathematical skills in children attending regular public schools.

As indicated above, the CHAT-based programs have a different conceptual approach to teaching math, compared to traditional math programs. That is, common educational programs used in Russian primary schools are based on the principles of traditional didactics first introduced by Komensky (Davydov, 1974). His fundamental concepts were as follows: the principle of awareness and activity, the principle of visibility, the principle of gradual and systematic learning, and the principle of repeated practice to master new knowledge and skills. In his analysis of the educational content of traditional didactics, Davydov pointed out the empirical generalizations that form its foundation. They form through observation and comparison of single objects, the definition of their common property, and their categorization into a group or class (Davydov, 2000). Methodologically, the acquisition of new knowledge is basically reproducing pre-set action pattern by the students, or remembering already established knowledge (Pavlova, 2008). Thus, the primary mathematics course is based on the conventional understanding of number as a common property of a set of objects, i.e., as a numeric characteristic of a group. For example, comparing particular objects (two apples, two pencils, two trees) students discover their common property, the number of two.

CHAT-based programs build the mathematics course on the basis of a theoretical concept of number as a relation between the measured magnitude (quantity, length, area, volume, etc.) and the measure unit (Rosas et al., 2017, 2019; Solovieva et al., 2016; Zárraga, Solovieva & Quintanar, 2017; Solovieva & Quintanar, 2022). Teaching methods are also fundamentally different. That is, in the CHAT-based programs, classes are organized around the solving of a specially selected problem, and corresponding tools are offered. Then, the selected method is modeled, and further, students and teachers work with the model created (most often, this model comes in the form of a number line).

Note that the two types of educational programs differ in the content used for the mastering of concepts and the ways to work with them. For instance, in the CHAT-based system, both arithmetic operations and textual problems are first solved by means of a model. Normally, it is a drawing or a scheme reflecting the key conditions of the problem. Therefore, the ability to write down and read the inscription of a number in decimal notation correctly depends on the understanding that number as a relationship of system of measures. Conventional programs, in turn, focus on the remembering of a particular algorithms and practicing the strategies of performing the actions. Much less attention is paid to the understanding of what is beyond them, and therefore, to the development of thinking. It was Vygotsky who first emphasized the vital importance of the content of learning and the mastered scientific concepts. He indicated that the consciousness and mastery of one's mental processes come "through the gate of scientific concepts" (Vygotsky, 1956, p. 247). Thus, what distinguishes the CHAT-based program from a conventional one is the content. In this regard, it is interesting to analyze if there are actual differences in the dynamics of formation of mathematical skills in children studying under a traditional program and CHAT-based one.

9.3 The Relationship of Executive Functions and Mathematical Skills

Multiple research studies revealed that the academic performance in mathematics is related to individual cognitive characteristics such as intelligence (Deary et al., 2001; Sternberg et al., 2001), number sense (Halberda et al., 2008; Mazzocco et al., 2011), speed of information processing (Semmes et al., 2011), and memory (Bull et al., 2008). Moreover, development of self-regulation is of great importance, particularly in primary school age (Jarvis & Gathercole, 2003; Bull, 2001). At this age, the child has to adjust to a new system of social requirements and rules set not only by parents, but also the teachers. Development of executive functions plays the key role in this process, predicting not only children's academic performance, but also their future well-being (Belyaeva, 2018).

In the assessment of executive functions, we relied on Miyake's theory (Miyake, 2000) because it was supported by previous research with a Russian sample of children (Veraksa et al., 2022). According to this theory, executive functions include three components: inhibitory control, working memory, and cognitive flexibility.

Inhibitory control is the ability to suppress the dominant/pre-potent response in favor of the less habitual behavior required to complete the task. Working memory is the ability to retain and manipulate information to solve various problems within a short period of time. Switching or cognitive flexibility is a characteristic that defines the ability to switch between different stimuli, problems, ways to solve them, and so on (Diamond, 2013; Duval et al., 2016).

Extant scientific reports present substantial data confirming the relationship between the components of executive functions and children's performance in mathematics. In particular, it has been shown that working memory contributes to the acquisition of mathematical skills (Bull et al., 2008; Kyttala et al., 2003). A meta-analysis of 28 studies involving children with math learning difficulties demonstrated that verbal working memory was the strongest predictor of mathematical performance (Swanson & Jerman, 2006). Further, researchers reported the relation between children's inhibitory control and the early development of their mathematical skills. Moreover, these relations persisted even when children's age, mothers' educational level, and children's vocabulary were taken into account (Espy et al., 2004).

The work by A. N. Veraksa and colleagues revealed that the level of executive functions was directly related to the children's ability to master new mathematical concepts (Veraksa et al., 2020). In this study, children took part in a learning experiment aimed at facilitating their developing understanding of the geometric concept of area. The participants were randomly divided to groups that received one of the three types of instruction: symbolic contextual, modeling, or traditional approach. The results demonstrated that the effects of these different approaches varied as a function of the child's level of executive functions. Students with low level of executive functions mostly benefited from the symbolic contextual approach (ibid.). Students with high level of executive functions showed better dynamics under the modeling approach. Students with a medium level of executive functions got better results with the traditional approach. This study confirms the significance of careful selection and design of individual educational programs taking into account the initial level of executive functions development in children.

The data obtained through longitudinal studies demonstrates that executive functions predict academic performance in primary school even when they are assessed as early as preschool age (Duncan et al., 2007). Further, executive functions are closely related to mathematical skills (Jarvis & Gathercole, 2003; Bull, 2001; Clements et al., 2016; Veraksa et al., 2022). For example, both inhibitory control and cognitive flexibility in preschoolers are predictors of mathematical skills later on (Best, 2011; Espy, 2004). Meanwhile, low level of executive functions is often associated with difficulties in math learning (Swanson & Sachse-Lee, 2001).

Meta-cognitive regulatory skills are similar but not identical to the executive functions. The former are skills of higher organization, and they include planning, monitoring of understanding, and evaluation. There are a number of investigations that examined the relationship of meta-cognitive regulatory processes and mathematical skills (Roebers et al., 2012; Veraksa & Veraksa, 2021). The study of the interconnections of planning, monitoring, and evaluation with mathematical skills and the primary and middle stages of educational system revealed that the students successful in mathematics had a higher level of meta-cognitive regulatory functions (Lucangeli & Cornoldi, 1997). However, the outcome of the examination of the relationship of such regulatory functions and mathematical performance is not homogenous. For example, the results of the multiple regression analysis held by V. I. Morosanova et al. where mathematical test performance was used as a dependent variable demonstrated that in school children, among 48 significant predictors of conscious regulation characteristics were not listed (Morosanova et al., 2017). Although the authors of that study built it based on their own concept of self-regulation, which warns us against drawing firm conclusions.

Therefore, existing research findings clearly indicate the importance of taking executive functions into consideration when studying mathematical skills in primary school children. Further, they suggest that children with different levels of executive functions may require different educational programs to show greatest benefits for mathematical learning.

The main goal of the present research is the evaluation of the contribution of the type of mathematical program to the growth of mathematical skills in 1st graders depending on their level of executive functions development.

9.4 Methods

9.4.1 Materials

Mathematical skills of the 1st graders were assessed in the beginning and in the end of the school year. For the assessment, those mathematical skills were selected, the development of which is given the most attention in the first grade. This is an understanding of the meaning of writing numbers and the "distances" between them (directly and using a numeric line), arithmetic skills and measurement skills. See a detailed description of the assessments below.

1. **Writing Numbers in the Base-10 System**. This task was used to assess children's ability to map discrete quantity onto a corresponding number written in the base-10 notation and vice versa. The assessment consisted of two parts; each part included 12 items. In the first part, children were shown a picture depicting a large collection of blocks: small ones (each representing a single unit) and long ones divided into 10 equal parts (and thus representing a collection of ten single units). Children were then presented with a written double-digit number and asked to circle the number of blocks corresponding to that number. In the second part, children were shown a picture with a pre-determined collection of blocks and asked to write the number corresponding to the depicted quantity. In half the items within each part, the ten-blocks were presented to the left of the one-blocks and in the other half the order was reversed. Children were given two minutes to complete each part. They received 1 point for every successfully completed

item. The maximum number of points that could be earned in this assessment was 24 (12 points per part).

2. **Arithmetic Skills**. This task was used to assess children's ability to add/subtract single- and double-digit numbers. The assessment included three parts. The first part consisted of 48 addition and subtraction problems with single-digit numbers. The second part consisted of 32 addition and subtraction with double-digit numbers (about half the items crossed the decade and the other half did not). Whereas in the first two parts, children had to find the sum or difference between numbers, the third part consisted of 48 addition and subtraction problems (single-digit numbers), where the sum or difference was known and children had to determine the missing component (e.g., a missing addend or minuend). Children were given two minutes to complete each part. One point was given for every correctly solved problem. The proportion of successfully solved problems was calculated separately for each part, relative to the total number of problems in that part.
3. **Comparing Numbers**. This task was used to assess children's ability to compare numerical magnitude and distances between numbers. Each item included three numbers (a triad) that were depicted within an oval frame; a target number was printed at the top of the oval and the two answer choices were printed at the bottom. The child's task was to cross out the number at the bottom that was closest in magnitude to the target number. The assessment included three parts; each part included 12 items. In the first part, the triads consisted of single-digit numbers (e.g., target: 5, choices 2 and 7) and double-digit numbers (e.g., target: 42, choices: 34 and 71). The second part consisted of 12 simple items with larger distances between numbers, making it easier to make a judgment and the third part consisted of 12 complex items tasks with shorter distances between numbers. Children were given 2 min to complete the whole task. Children received 1 point for every successfully completed item. The maximum number of points that could be earned in this assessment was 36 (12 points per part).
4. **Measurement**. This task was used to evaluate children's ability to measure objects with a given unit of measurement. This assessment consisted of eight items where the respondents were asked to find the area of a rectangle by using a depicted measurement unit. The rectangles were drawn on graph paper. There were four pairs of items that differed both in the size of the target rectangle and the measurement unit. Within each pair, one item depicted a rectangle with a grid present (which can aid in determining area) and the other item depicted a rectangle without a visible grid. For each item, participants were asked to write the number of units that comprised the area of a given rectangle. Children were given 2 min to complete the task. They received 1 point for each successfully completed item. The maximum number of points that could be earned was 8.
5. **Number Line**. This task was used to evaluate children's ability to estimate numerical magnitude by placing a given number on a number line. The assessment consisted of 8 items, which differed in several ways: the size of a single unit marked on the number line, the direction (with the zero point being on the left or the right end of the number line), and the presence/absence of other numbers on

the line. Children were asked to mark the required number on the line. Children were given 2 min to complete the task. They received 1 point for each successfully completed item. The maximum number of points that could be earned was 8.

Executive functions were assessed with a toolkit that included tasks measuring visual and verbal working memory, inhibition and cognitive flexibility.

(a) Visual working memory was assessed using the NEPSY-II «Memory for Designs» subtest (Korkman et al., 2007). Each child was presented with a field with cards for 10 s. Afterward, the field with the cards was replaced with an empty one, and the respondent was supposed to pick the right cards from the available set (which included distractors) and put them in the same positions they were on the sample field. The task included 4 trials, with the number of target cards increasing across trials from 6 to 10. On each trial, children received points both for the accuracy of content (choosing the correct cards) and spatial (placing the selected card in the correct location). Bonus points were given if the correct card was placed on the right place. Total score for this task was obtained by adding all the points; the maximum number of points that could be earned was 150.

(b) Inhibitory control and cognitive flexibility were assessed using the NEPSY-II Inhibition subtest (Korkman et al., 2007). This assessment included a series of 6 tasks. Two of them required naming figures and directions of arrows as presented on the stimulus sheet, while the other two required doing the opposite (e.g., if an arrow is pointing up, the child is supposed to say it is pointing down). The last two tasks focused on the switching between naming figures/directions either correctly or in the opposite way, depending on the color (e.g., if a white arrow points up, the child is supposed to say it points down, but if a black arrow points down, the child is supposed to state that correctly). The response time and the number of errors, either self-corrected or not self-corrected by the child, was recorded. Complex total scores for naming (based on the first two tasks), cognitive inhibitory control (based on the next two tasks), and cognitive flexibility (based on the last two tasks) were calculated; with the maximum number of points being 19.

(c) Verbal working memory was assessed using the Wechsler's Digit Span task (Veksler, 2003). It consists of increasing spans of digits that are to be repeated, both in the direct and reverse order. The experimenter read the digits at a steady pace and the child was supposed to reproduce it. Depending on the size of the number set, a respondent could obtain 2–9 points. The total score for the verbal working memory task was calculated by adding the points across trials. Separate scores were computed for direct- and reverse-order repetitions.

9.4.2 Sample

The sample consisted of 179 1st grade students from Moscow public schools (53.6% boys), of whom 81 (45.3%) children followed a CHAT-based program, and 98 (54.7%) children followed a traditional program. The evaluation of the students' distribution by gender between the two programs did not show significant differences (Chi square = 1.15, $p = 0.284$).

9.4.3 Procedure

The study took place during the school year in 2020–2021 and included three stages. In the beginning of the school year (October 2020), the students were tested to assess the initial level of mathematical skills. At the end of the school year (April–May 2021), the same assessments were conducted to determine the growth of mathematical skills over the course of the school year. All assessments were administered in a group format (to the whole class). Assessment sessions lasted 40–45 min, which is the length of a standard lesson. Following the second group assessment of math skills (at the end of school year), individual sessions were conducted to evaluate executive functions of the study participants. The assessment of executive functions was carried out in one meeting lasting 35–40 minutes in a quiet separate room of the school where first-graders studied, by specially trained testers.

9.5 Results

The results were evaluated for the individual and joint contribution of the two key factors—the level of executive functions and the type of the educational program–on the dynamics of mathematical development in primary school children.

9.5.1 Assessment of the Relationship Between Executive Functions and the Dynamics of Mathematical Skills Development in Primary School Children

The sample was divided into 3 groups (k-means cluster analysis) with low ($N = 44$), medium ($N = 50$), and high ($N = 54$) level of executive functions development. These groups differed significantly by all the parameters except verbal working memory (Kruskal–Wallis H test).

Our evaluation of the students' distribution among programs depending on their level of executive functions development didn't reveal any significant differences between the subsamples (Chi square = 2.62, $p = 0.269$). The analysis of differences (Mann–Whitney U test for independent samples) confirmed the absence of a significant relationship between educational programs and the level of executive functions under the current study as of the end of the school year ($p > 0.05$).

To assess the overall contribution of the executive functions to the formation of children's mathematical skills, Welch's one-way ANOVA was performed. It revealed that the students with different level of executive functions demonstrated significant differences in their ability to write numbers in Base-10 system ($p = 0.047$) and perform measurement task ($p = 0.002$). For example, the children with low and medium level of executive functions demonstrated weaker dynamics in performance in writing numbers than the students with a high level of executive functions (Games—Howell). As for the measurement task, the most noticeable dynamics in the performance were demonstrated by the students with high and medium level of executive functions compared to the participants with low level.

The contribution of individual parameters of executive functions to the dynamics of 1st graders' mathematical skill development was assessed by running several multiple regression models. In these models, executive functions acted as predictors and mathematical skills served as dependent variables (Durbin-Watson test $\in$ [1.5; 2.5]). It was revealed that the better visual working memory of the 1st graders was, the more their performance in writing numbers in Base-10 number system improved (predictor's contribution = 8.4%). The improvement in the measurement was also greater in children with higher levels of visual working memory (predictor's contribution = 8.8%).

9.5.2 Assessment of the Relationship of Educational Programs in Mathematics and the Dynamics of Mathematical Skill Development in the First Grade

The sample was divided in two groups: the participants who studied under CHAT-based program ($N = 81$) and those who followed the traditional one ($N = 98$). The development of mathematical skills across the two groups was compared using the Mann–Whitney U test. It revealed that the students receiving a CHAT-based program showed a trend for better performance, compared to those receiving a traditional program, in the development of three mathematical skills: writing numbers in Base-10 number system ($p = 0.070$), addition and subtraction of double-digit numbers ($p = 0.074$), and operations with a number line ($p = 0.071$).

9.5.3 Assessment of the Joint Contribution of Educational Programs and the Level of Executive Functions on the Dynamics of Mathematical Skill Development in Primary School Children

In order to evaluate the joint contribution of the factors of the program and executive functions on the development of mathematical skills, a number of general linear models were built (ANOVA). No such contribution was registered. In fact, children with different level of executive functions from CHAT-based and traditional programs master mathematical skills equally. The above-mentioned factors make a more noticeable contribution separately.

In addition, we decided to examine the relationship of the level of executive functions and the level of development of mathematical skills of children studying under each type of program.

The following correlations were discovered (Spearman's ρ test, $p < 0.05$) for the 1st graders **studying under a traditional program**:

- The ability to write numbers in Base-10 system was better developed in those children whose visual working memory was better than their peers,
- The development of the ability to add/subtract single-digit numbers was better in children with higher level of cognitive flexibility,
- The ability to compare double-digit numbers was better developed in children whose mental processing speed was lower,
- The measuring abilities were better in 1st graders whose working memory was more developed and mental processing speed were higher.

Only two significant correlation (Spearman's ρ test, $p < 0.05$) was revealed for the 1st graders **studying under CHAT-based programs**:

- The ability to add/subtract single-digit and double-digit numbers developed faster in children whose mental processing speed was higher,
- The ability to operate a number line was better developed in children with lower level of cognitive flexibility.

The executive functions components produced more correlations, and in this regard, perhaps they are more important for the students studying under traditional program than those following a CHAT-based one.

9.6 Discussion

This paper describes the study of the influence of two factors on the dynamics of mathematical skill development in 1st graders. The factors are: (1) the level of executive function development and (2) the educational program the children followed. The obtained results revealed that the higher the level of executive functions, the better the dynamics of development of mathematical skills such as writing numbers in Base-10 number system and measurement. This data coincides with the outcome of other research works (Bull & Lee, 2014; Bull & Scerif, 2001; Lan et al., 2011; Lubin et al., 2013; Passolunghi & Cornoldi, 2008). However, our study didn't reveal any correlation between the general level of executive functions and the dynamics of development of mathematical skills such as arithmetic operations, working with a number line, or comparing numbers. Yet, interestingly, the number of correlations involving the components of executive functions turned out to be related to the type of educational program the children followed. For CHAT-based program, there were fewer connections. Only the mental processing speed and cognitive flexibility were connected to a more successful mastery of mathematical skills. Additionally, the lower the cognitive flexibility was, the better the ability to work with a number line was developed. This phenomenon is not easy to explain (it might be a research artifact). In case of traditional programs, apart from the mental processing speed and the cognitive flexibility, it was visual working memory that contributed to the development of mathematical skills. This could be explained that traditional teaching model relies on children's abilities (including self-regulation) and their initial level to a larger extent than CHAT-based programs. The latter are focused on the zone of proximal development and appeals to the functions that are not yet developed but are in the process of formation and maturation (Tsukerman, 2010; Vygotsky, 1956).

On the other hand, our study revealed that CHAT-programs only show a tendency to higher dynamics of mathematical skill development. Nevertheless, more such skills are affected than in the case of a traditional program. These programs include writing numbers in Base-10 number system, addition/subtraction of double-digit numbers, and finding the correct position of a number on a number line. It is possible that only a mere tendency was registered because our assessments (level of mathematical skills development in the beginning and end of the year) only took place in the first grade, and only five months had passed between the two assessment periods. A longitudinal study could provide a more precise answer to this question.

In our evaluation of the joint contribution of executive functions and educational program, it was discovered that the 1st graders with different level of development of executive functions studying under different programs master mathematical skills equally. Thus, it is not valid to assume that one program is more suitable for children with a certain level of executive functions development. Independently of executive function development, and of the program, children advance at their own pace. This fact could be of high practical importance. Perhaps, educators should not focus so much on the selection of children for a particular program based on their self-regulation level, but shift to the selection of more efficient programs and the quality of their implementation.

The results of this study can be interpreted from two perspectives. Well-known works by L. S. Vygotsky dedicated to the analysis of the relationship of learning and development (Vygotsky, 2004; Elkonin, 1966) emphasize an approach suggested by J. Piaget. According to Piaget, educational programs should be adjusted to the child's level of development. In this regard, we should note that nowadays, it is common to choose an educational program in mathematics to match the child's executive functions level, as more advanced and complex programs require better executive functions. The present study did not reveal any correlation between the level of cognitive regulation in children and their performance in different programs. There were no significant differences in the learning dynamics. On the other hand, Vygotsky believed that educational programs should be built in such a way that learning entails development. This can only happen in the case of modification of learning content, i.e., the concepts and the problems students are to master and solve. In our study, the role of content did not manifest as we expected which can possibly be explained by the short period our participants spent at school. Validation of this assumption will become the goal of future research.

9.7 Conclusion

This study revealed that the higher the level of executive functions was, the better first-grade children's performance in mathematics was. However, depending on the educational program, the contribution of executive functions might differ. The individual components of executive functions demonstrated more correlations with the mathematical skill development in children studying under a traditional program than in those following a CHAT-based program. The comparison of efficiency of the programs brought ambiguous results. Still, there are certain grounds to consider CHAT-based program as more efficient. The obtained data raised the important question of the relationship of learning and development. Should we understand the child's level of development as the main factor of learning efficiency? Or should the content of educational programs be modified so that leaning would "indeed lead the way for development"?

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Chapter 10
Scaffolding Multiplicative Concepts Formation: A Way of Digital Support

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Abstract The balance scale problem is a widely known material for studies on the development of thinking. It is hard for students to master the multiplicative concept of equilibrium, even though the general idea "the further from the fulcrum – the heavier" is commonplace and students are often provided with real or digital simulations of the lever – so they could explore it as much, as they need. The psychological researches since Piaget regard the concepts, based on logical multiplication (i.e. the consideration of the changes of two parameters) as those, which demand acquisition of special intellectual operations by students. As we examine the early stages of multiplicative concepts' formation within the activity approach framework, we include special modelling means to support students' own inquiry in order to promote their actions to the conceptual level of orientation in the coordination of two magnitudes' changes. Thus, students' manipulations with weights – adding, moving and removing – become mediated by the specially organised additional work: the evaluation of the contribution of each weight unit to the total load on each side of the lever. Within this approach, students are able to start from the trickiest tasks with several weights located in different positions (which were considered to be exceedingly difficult since Siegler's studies) and the concept is acquired from their own actions mediated with this "additional" modelling. The digital support design, thus, has to serve several purposes, among which most important is to scaffold problematization, modelling and distribution of operations among partners. Though many modern lever simulations provide students with functionality enough to test different combinations of weight units, they do not provide the educational designers with enough ways to challenge and test students' modelling. The educational module on the concept of balance, which we have designed based on the principles of the learning activity theory, shows, that a significant number of functions has to be added to the computer simulation of the lever to support the tasks modifications, which will make students rely on modelling.

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10.1 Introduction

We address the problem of concept formation within the cultural-historical and activity approach framework, which considers concepts in their relation to actions responsible for concept emergence and concept functioning. Since Vygotsky (1986), the "scientific" (school) concepts formation, as opposed to the acquisition of "everyday" concepts, has been a challenge. Galperin and Leontiev made further steps on the way, that Vygotsky has outlined, and connected the acquisition of new concepts to the new actions, which are to be mastered by students in situations which they had never confronted in their everyday life. These tasks demand to act conceptually (Leontiev, 1978), as opposed to "natural" behaviour patterns, prompted by situations' vivid characteristics and "common sense" notions. According to Galperin (1989), the ability of students to perform their own actions according to the concept allows them to overcome vivid prompts, thus, such tasks are critical for the assessment of concept formation. He described the role of special "orientative" actions in concept formation and the role of concepts in the development of these actions as orientative (Davydov et al., 1983; Engeness, 2021; Galperin, 2007; Galperin & Talyzina, 1957; Talyzina, 1968).

These actions are defined as "matter"-oriented (Davydov, 1990), which means, that one has to operate the matter, the objects, according to the concept. At the same time, Davydov's idea was to introduce students to the special concepts' origins within human cultural history through these actions: "In the process of mastering, the content of these forms of social consciousness as a product of the organised thinking of many generations of people (more precisely, of their theoretical thinking), there emerges in the child a relation to reality that is linked to his development of theoretical consciousness and thinking…" (Davydov, 2008, p. 57).

According to Davydov's theory and the cultural-historical approach in general, learning activity is referring to concepts while reconstructing the contexts of their origins. "…The content of learning activity is theoretical knowledge…This hypothesis as to the content and sense of learning activity is based on a special study of the history of public education together with its current trends" (ibid., p. 115). The appropriate unit for educational design here is the learning task, wherein "the school children first master a general method for solving particular tasks" (ibid., p. 124).

As well as other researchers following Davydov's approach (Aidarova, 1978; Coles, 2021; Repkin, 2003; Rubtsov et al., 2021; Zuckerman, 2020; etc.), we find it productive and prospective and our studies, including current one, are being made within this framework. The developmental instruction approach demands that we analyse the concept's emergence and evolution within meaningful cultural activity and reconstruct the situation of the concept origin in students' own actions aiming to provide the essential modelling (orientation) means of actions alongside with the main meanings, twists and intrigue of the matter, which are essential from the psychological point of view.

To examine the mechanisms behind concept formation, we design the exact way for the desired concept's acquisition (the sequence of actions, their content, material, means, etc.) and conduct an experimental teaching series to see whether the designed formation procedure will provide our participants with the concept, which will work as planned, – according to Vygotsky's "experimental-genetical method" Galperin's formation procedure (Liders & Frolov, 1996).

The challenges and benefits of digitalisation are considered within the learning activity framework (Talyzina, 1989). Implementing activity approach ideas to computer support design demands logical and psychological analysis of the task domain under consideration. The initial relation of objects, which is at the core of concept formation, is thus revealed and presented in the set of learning tasks and in specially organised interactions in the learning activities.

The question to be answered is: "What functions of support towards students' own actions may be transferred to the computer, given that the computer's part has to be meaningful, rather than formal, and beneficial in terms of resulting concept-mediated thinking?".

10.2 Multiplicative Concepts Formation

The concepts based on logical multiplication (Inhelder & Piaget, 1958) involve changes of two independent parameters, which have to be considered simultaneously, as their coordination matters. There is a great number of tasks, which imply "multiplication" in this sense: for example, including objects to some sets and subsets by their characteristics, comparing the amount of liquid in two vessels of different height and width, comparing areas of different shapes, etc. Among these tasks, the most outstanding for many reasons is the balance problem: it is most naturally set, it refers to everyday experience, it deals with parameters of different measures, it has many task variations, it is interesting, the result is vivid, etc. The balance problem was hence one of the most popular "sample" problems for psychological investigation since Piaget.

It has less analogues in other contexts, which could address students to the pure "multiplicative" concept either for diagnostical purposes or in order to teach students proportional reasoning in case of an inverse ratio. Simple tasks, like inverse relations between a vessel's height and width, or the lengths of rectangular sides, are mastered by primary and secondary school students. Yet, some balance problem tasks remain difficult even for high schoolers.

Following Siegler's studies on the psychology of these concepts formation within the balance problem context (Siegler, 1976, 1978), there have been many researches, which had discussed in detail the strategies and rules, which students (since kindergarten till adults) come up with, as they try to predict, which side of the lever will overweigh the other (Boom et al., 2001; Filion & Sirois, 2021; Hofman et al., 2015; Jansen & van der Maas, 2002; Li et al., 2017; Martin & Rubtsov, 1991; Siegler & Chen, 1998). Students experience well-known difficulties, when two magnitudes weight and distance are obviously working against each other to make the other side "heavier" (as on Fig. 10.1a). Mastering and applying these strategies may help to

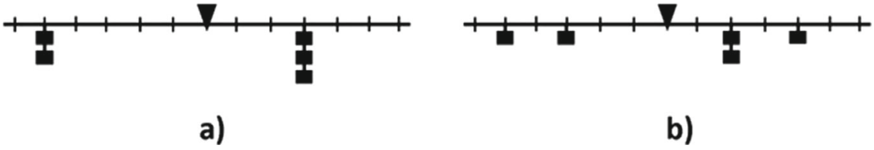

Fig. 10.1 Critical tasks: "What side overweighs the other?" **a**) "conflict" between weight and distance; **b**) "scattered-weights" task

predict, which side will go down in simple cases, yet other tasks, involving, e.g. "scattered" weights (weights in several locations on the lever-Fig. 10.1b) will still be unsolvable for students of all ages (Siegler, 1978).

These tasks are critical as they challenge students' comprehension of the matter.

Most students grasp the idea that besides the amount of weight units their position should be taken into account (units closer to the edge are "heavier" than those closer to the centre), yet no one can invent the "balance law" himself. Thus, an obvious way to deliver the balance rule is (1) to present the balance problem to students in order to make them interested, (2) allow them to work with the lever in order for them to grasp the general ideas: "weight and distance matter", "the farther – the heavier" and (3) finally, as students fail to discover the exact way to coordinate the parameters, their teacher delivers them the formula: the equivalence of multiplied weights and distances, which stands behind the equilibrium.

Eventually, the computer support may play its part as the motivator and the simulator. The lever can be presented on the screen and provide "users" with opportunity to attach weight units or objects of different masses to different locations along both sides of the lever and to observe the resulting balance state. The computer simulation is better than the actual lever, as it is precise and is not subject to real-life inconveniences of use. Its major role is to replace the complicated real-life objects with their model, where the magnitudes and their relations stand out more vividly: weight is presented with weight units, distances are marked with steps. Currently there are several software products, which provide the tasks based on the simulation as described and studies, which use digital support for the balance beam problem (Hofman et al., 2015; Tahai et al., 2019; Van der Graaf, 2020).

The design of computer software support inevitably relies on the authors' comprehension of the psychological mechanisms behind students' acquisition of concepts, the multiplicative concepts in particular. The majority of computer simulations are aimed to lead students to formulation of "rules" and "strategies" and scaffold students' promotion through practical trials and feedback, while they learn to choose and apply the strategies, which they had retrieved from their own experience, in each subsequent task with increasing difficulty level (Hofman et al., 2015). In particular, the similar "trial-feedback" approach was also used to teach computer (AI) to solve balance scale tasks (Sage & Langley, 1983). In Tahai et al. (2019), the additional functions of computer support were to monitor students' success and concentration level and adapt the difficulty of each subsequent balance problem task in order to keep students "in the state of flow" during the whole "Scalebridge" educational game designed to teach proportional reasoning.

To our mind, the deficits in students' comprehension are rooted in the very content of students' actions while they are being taught. In this respect, we may as well rely on the studies mentioned above, as all of them vividly demonstrate the limitations of students' own possibilities to retrieve the balance law from their own manipulative experience with the lever (real or digital): the exact way of coordination between two parameters cannot be invented by students—especially in "critical" tasks.

The application of a consistent concept often differs significantly from its origins – and it is the latter that we have to reconstruct for students. The demonstration of the way the concept of equilibrium is used for solving balance problems nowadays does not allow students to grasp the orientation procedure behind the "short-cut" calculation algorithms, which deal with the two parameters directly. Frequent typical mistakes, which students make, when they try to learn from the reduced and shortened empirical strategies, show us that the origin of the concept is not there. Those who can solve any balance problem and use these strategies consciously and adequately have something to their orientation that is not externalised in the common problem-solving, but it comes to light, as we try to reconstruct the concept's origin in students' actions. Only as we analyse the most critical tasks (the "scattered-weights" situation in particular) and try to extend the orientation needed, we come to the necessity of special actions: the evaluation of the third magnitude (the "load"), which is equal, when the lever is balanced – and thus, stands for the equilibrium itself. These actions have a purely orientation purpose, as the load cannot be dealt with directly in reality: we can operate the load only through coordinated changes of weights and distances. Thus, to solve critical balance problems, students have to evaluate the contribution of each weight unit to the total load on each side of the lever – be it the current configuration or the result of students' possible operations.

Based on this assumption, a special learning module with appropriate computer support was devised and tested. In this paper, we will focus on some features of computer support, as our aim is to show that from the stance of Davydov's theory and the studies of his followers, the role which a computer may play, becomes larger, more detailed and benefits the results more, than digitalisation within traditional didactics. Thus, as we present the way to form the concept of equilibrium, we will pay special attention to these questions:

What computer support will be adequate for the learning task, which we design?
What computer support will help students rely on the concepts as quintessence of the cultural ways of handling the matter?
What objects and their characteristics should be emulated through a computer and what operations with objects are to be made available through computer support?

10.3 The Developmental Computer Support for Balance Concept Formation

The balance problem allows us to illustrate the collision between two didactic approaches: “formal” and “meaningful”. There is a “formal” rule and an appropriate formula that support simultaneous consideration of two parameters and direct calculations. However, the rules of balance, being simple to use, set significant limits on the variety of tasks, which students can solve. On the other hand, “meaningful” representation of the balance problem through modelling addresses students to the variety of tasks that are not solved at a glance by simple formulas, and usually prompt trial-and-error method. These tasks exclude “work-around” ways and require referring to the “invisible” third magnitude. Students have to divert their attention from salient magnitudes, that are observed and described through numbers and well-known formulas, towards their new “meanings” brought by their role within the coherent view of the task situation in terms of “load”. The appeal to the third magnitude’s changes allows to anticipate the “theoretical” (conceptual) image of the final result, towards which the future operations on the weights will be adjusted. Thus, students are put to use something that cannot be pointed out within the object they are dealing with: reasoning and inquiry are done with “additional” modelling means, which do not provide an answer immediately and directly—and that is what conceptual thinking is about.

The basic learning problem within the developmental instruction framework (Davydov, 2008) should exploit those tasks, which cannot be solved by students effortlessly from the start. Unlike traditional didactics, there will be no moving from simple to complex here, nor there will be any immediate problem-solving through quick guessing, or trial-and-error method. Students’ “inquiry” on the conditions of changing and preserving equilibrium starts from testing the lever with complicated weights’ configurations, which are already balanced in an unobvious way. The general task for students is to find the way to compensate the changes, brought by “a partner” on one of the lever sides, with their own operations with weights as a response. Task should make “direct” compensations not possible, as these solutions are both evident and do not reveal the actual problem of equilibrium. Students have to find as much ways of regaining balance, as possible, and at the same time pay special attention to situations, when the changes made by one “player” cannot be compensated in any way by his partner.

At this point, the computer support has to provide the simulation of the lever so that students will be able to move or add weights to both sides and to check the balance. Most simulations, which present the balance scale and the weight units, will be sufficient enough for setting the general problem: “the naïve suggestions will not work”.

Students’ actions (moving, adding or taking away weight units) reveal the “relativity” of the “heavier” and “lighter” notions in their “routine” sense: the real outcomes do not seem to rely on the weight solely. The unique connection between weights and their distances deserves the emergence of the special magnitude responsible directly for balance. Thus, the learning problem introduces new coordination means, which allow to evaluate the results of would-be operations with weights—thus both partners will be able to plan their own "movements" simultaneously, irrespective of the past operations and their partners' exact final configuration.

We are to provide the space and instruments for students' own inquiry, which is exactly what the learning activity is about, where the functionality of special procedures and modelling means will be discovered and reconstructed to serve successful problem-solving. A challenging task is needed: the one, which cannot be solved with "work-around" speculation or "short-cut" techniques. The "scattered-weights" tasks differ from other tasks significantly, as they cannot be solved with self-devised strategies and adults are no better than school students in this respect. Only concept-mediated comprehension of the equilibrium law will help to estimate the balance in these tasks. Thus, we choose the typical scattered-weights tasks as our main setting: "Balance three weight units on the left side with two weight units on the right side" (Fig. 10.2a). Moreover, the intrigue of the learning task, as we have designed it, is to make the three weights, already attached to different positions on one side of the lever "by someone", balanced with another two on the other side (Fig. 10.2b).

The "discrepancy" between "controllable" parameters and "uncontrollable" outcomes revealed through experiments denies the "absolute" role of weights' number and position and sets the question about the nature of their connection. What is "equal" in "equilibrium"? If it is neither weight, nor distance, considered separately, or some proportions of these standalone magnitudes, then what is it? The special notion of "load" and the corresponding "load-measures" play the "mediation" role: they help to grasp this relationship apart from the particular weights' configurations through appropriate evaluation procedures in the separate modelling space (Fig. 10.3).

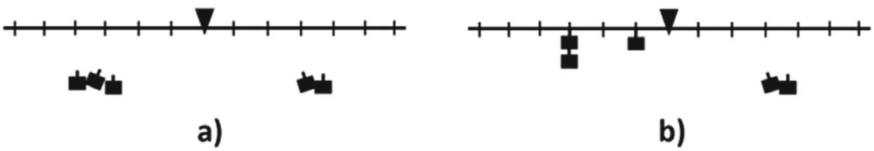

Fig. 10.2 The setting for the learning problem

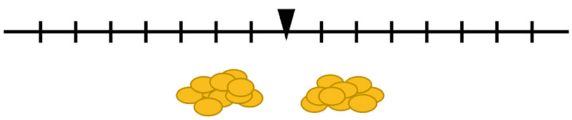

Fig. 10.3 The total value of loads on both sides of the lever or the total load do not depend on the exact configuration of weight units, which create the load. In case of equilibrium the load has to be equal for both sides

The work within modelling space is directly related to the setting and solving of the learning problem. Special symbolic means are designed to scaffold students' orientation: tokens or "coins", which are used to evaluate the balance directly according to the position of each weight unit. A certain number of them is to be attributed to each weight on every side of the lever to assess its contribution to the equilibrium of the whole lever (see Fig. 10.4). One "coin" is "paid" for every single "step" of the weight unit from the fulcrum.

The equivalence of the composed loads for both sides of the lever sets the requirements for equilibrium and eventually the pattern for solution design in the most generalised way. The positioning of any number of weights is thus guided by the required load (Fig. 10.5). The configuration of weights is determined both: in "conceptual" sense by the load, which is collected from the contribution of each weight, and in the "operational" sense by the particular limitations of a task (the already attached weights, the available weights' positions, etc.).

The equilibrium itself thus gains its measure and becomes a magnitude: if the number of coins for each side is made the same, the lever is balanced. This magnitude may be named as "load", and it is a pre-concept for the "torque". It is an intensive magnitude and students cannot operate it directly. Only by changing the weights and their distances can they alter the load. Unlike solving an inverse proportion between weight and distance, the evaluation of load is an extended procedure, which brings in front the "third value" behind the establishment of equilibrium, while the latter is never indicated in simple formulas for the balance law.

The "coins" serve orientation purposes only – they are not objects, which can be hung on the lever, nor do they tell directly what is to be done with the weights: adding one coin does not necessarily mean adding one weight, etc. Yet any successful "moves" performed on the lever by students happen only as students evaluate the result of their future actions in terms of load, and then select their operations with the weight units – adjust them to get the needed load. The operations needed to solve the problem do not coincide with the actions responsible for the orientation itself. At the same time, the coins perform the "control" functions towards the problem-solving

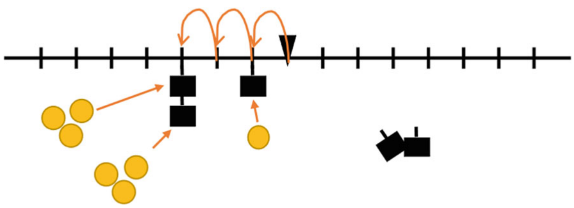

Fig. 10.4 The introduction of the special means "coins" to assess balance directly

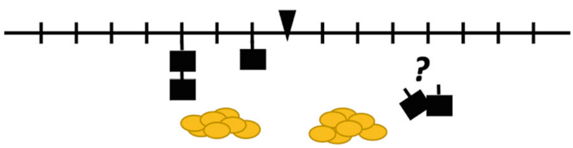

Fig. 10.5 The load-tokens allow to control the load, which will be created by the weight units on the right side

procedure even before the solution is checked with the lever (real or digital one). However, as our previous researches show (Vysotskaya et al., 2022), students tend to avoid the application of these essential modelling means – thus, the appropriate educational module and learning materials should be carefully designed.

This is the key part of the concept formation – the "seemingly" unnecessary excessive actions of attributing tokens to each weight to evaluate its load, actually provide for students' ability to solve most difficult "scattered-weights" tasks on their first attempt. At the core of the learning task are students' attempts to coordinate the changes, which they perform on the balance scales (adding, moving or taking away weights) and their own work with these symbolic means, which address the load directly. Only as students overcome the "natural" way of treating the load-tokens as weight units and adjust their operations with tokens so they would serve as compound measures of load and predict the equilibrium – only then these symbolic means gain orientation functions and their actions become concept-mediated. Eventually, a student learns to "see" the possible changes in load behind each moving and removing of a weight unit, which he or his partner is performing. Natural objects and operations are thus seen through "conceptual" lens.

The computer thus has to scaffold these new modelling procedures. If tokens are displayed within the lever simulation and allow to add and move them freely, then students can perform load-evaluation on the screen, which especially benefits the distant-learning process.

Aside from adding special objects ("coins") to the lever simulation for modelling purposes, it is far more important for computer support design to scaffold proper concept formation by creating conditions, which keep the intensity of orientation throughout the tasks. Before the basic conceptual orientation procedure (the evaluation of load) is interiorised, students are to master it through a series of tasks, modified in various aspects (Engeness, 2021; Galperin, 2007). These modifications are aimed to prevent students from "cutting" the orientation procedure short and skipping some steps, or referring to some "work-around" ways to solve the problem. Thus, we chose those modifications, that make the tasks unsolvable, if students do not consider the changes of load, which each "movement" brings. Each time a student solves the task, he should be driven to refer to evaluation of load in the modelling space – otherwise the "conceptual" means of solution will not be adopted: during this early period of concept formation once a task allows to solve it without "coins" students will surely skip the "bothersome" procedure.

Among these modifications are the limitations on the operations available for students: limited number of weights, the necessity to put all given weights before testing the balance, the already attached weights, which can't be moved, cancelling some of the locations for weights, etc. (Fig. 10.6). In this respect, a computer simulation is vital, as it can literally cancel some opportunities for students, whereas a teacher can "announce" these restrictions for students when dealing with a real lever and try to control every student, but rule-violations can never be totally excluded.

Some tasks are specially designed to be unsolvable – students are to prove, that there is no solution (Fig. 10.7). The students, who were taught according to the educational module, which we developed, answered without hesitation and doubt

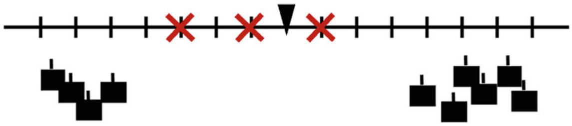

Fig. 10.6 The example of the task modifications: only one probe is allowed, all ten weight units have to be placed on the lever, some locations are disabled

that the balance cannot be achieved as the load on the right side is ten "coins" and the load on the left side is twelve "coins": "We can add only one "coin" by moving one of the weight units as far as possible, but to balance the other side we need to add two "coins"!".

Another example of task modification is the introduction of "unknown" weights, which have to be balanced with usual weight units. The first "screen" (Fig. 10.8a) shows the picture of a lever, which is already balanced with the combination of usual weights and a new unit of "unknown" weight. Then students are asked to balance on the first try another lever (Fig. 10.8b) with a preset number of usual and new weights, so that it is impossible to repeat the initial configuration. A more complicated version of this task demands, that the lever on the first "screen" has to be balanced before students can proceed to the final part.

In order to balance the lever (Fig. 10.8b) on the first try students have to "weight" the "unknown" unit. This can be done, using the previously balanced lever on the picture provided (Fig. 10.8a). Students are to evaluate the total load on the left side of the lever and get 12 load-tokens. Thus, the right side should also have 12 load-tokens: six of them are "produced" by the common weight unit on the sixth position, and the rest six tokens should be attributed to the "unknown" weight. A usual mistake is to suggest, that the weight of the "unknown" unit equals six common units. In this case, the attempt to use this "value" for balancing the new set of these weights on

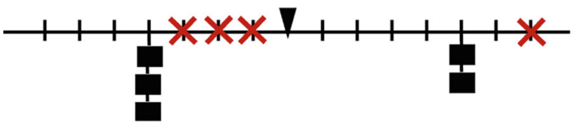

Fig. 10.7 The example of an "unsolvable" task: "*Decide whether the lever is balanced. If not, restore the balance by relocating only* ***one*** *weight unit* ***on the right side****!*"

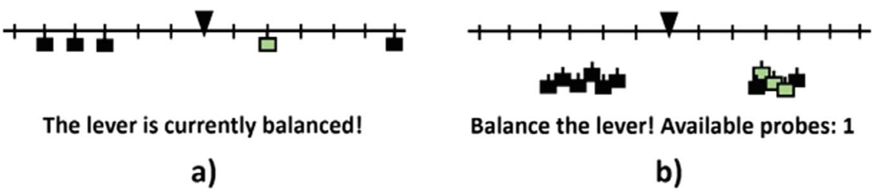

Fig. 10.8 The example of the "unknown" weights task

the lever (Fig. 10.8b) will fail. The right "value" is derived, if the "steps" made by the "unknown" weight from the fulcrum were taken into account: two steps from the fulcrum means, that the new weight "was made" two times "heavier" than his "real" weight. Thus students will be able to balance the final lever counting each of the new units as three usual weights (Fig. 10.8b).

The collision between the seemingly "natural" way to handle the matter and the "conceptual" way to solve problems comprises the most essential "psychological" task modification (Galperin, 2007). Tasks of this kind are especially important for diagnostic tools design to assess students' mastery of the load concept in solving balance scale problems (Fig. 10.9).

This task is most sensitive towards conceptual mediation of problem-solving, as the naïve immediate strategies are quite tempting here. The idea to add one weight to each side (as Mike suggested) naturally seems right, as well as the idea to move all weights one step apart (Nancy's variant). Yet both suggestions fail: if we evaluate the load for each side of the lever, it becomes obvious that the balance is ruined in both cases. Nevertheless, those students, who have acquired the concept successfully, experience no difficulties: they evaluate the load and find that only Pete's combination is balanced, though it may not look like it at first glance.

An important approach, that also considered the problem of multiplicative concepts formation, was developed upon further analysis of the content of operations, which correspond to logical multiplication. The difficulties in solving "critical tasks": conflict-weight, conflict-distance, conflict-balance conditions (Filion & Sirois, 2021; Hofman et al., 2015; Siegler, 1976) – happen, when students have to perform some special actions over direct operations with objects, rather than over the objects themselves. A way to approach this complicated system of cognitive actions was to distribute the essential parameters of the objects among partners, so that the

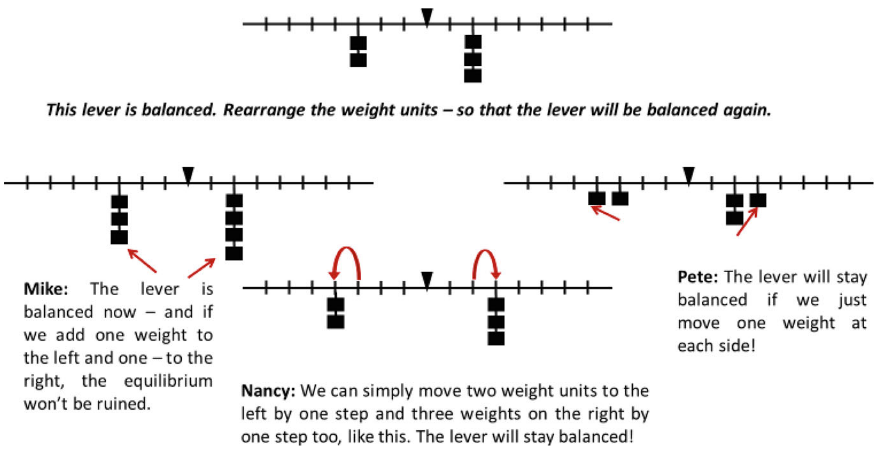

Fig. 10.9 The critical task: "regaining balance". The task goes as follows: "Students tried to achieve balance with the same number of weight units (three on the left and two on the right), but with another combination. Read their argumentation and choose the student(s), whose variant is balanced indeed"

result would be achieved only through appropriate coordination between students. This approach produced a number of researches on joint actions, which aimed to assess and study the ways of solving problems, in particular those based on logical multiplication (Davydov et al., 1996; Konokotin, 2019; Martin & Rubtsov, 1991; Rubtsov et al., 2021; Ulanovskaya & Rivina, 2018). For a number of multiplicative concepts (Rubtsov et al., 2021), it has been shown how the means of between-students coordination of their partial actions evolve to coordination of simultaneous changes of two magnitudes. Rubtsov (1987) describes the significant components of joint actions' organisation and underlines the most important psychological means to scaffold the distribution of actions: communication and reflection (Rubtsov, 1987, p. 33). The "objective" distribution is necessary here, as it allows a student to imagine and to analyse his own contribution to the object's changes with respect to his partner's "part"—thus, one derives and adopts the corresponding multiplicative concept from this very reflection of the coordination's basis and the comprehension of the concept's role in the orientation of one's own operations. Special modelling means are introduced to scaffold the concept's "work" in the orientation procedure (Vysotskaya, 1996).

The computer support of the appropriate organisation of joint actions thus becomes vital for our educational design. For our balance scale module and computer software, accordingly (Fig. 10.10), we have chosen two variants of distributed operations: either each partner is responsible for one of the lever sides, or one of the students can choose the location, and the other - can add or remove weight units. The work with the load-tokens in the modelling space thus comes forth as the space for negotiation between partners, as they try to coordinate their future partial operations on the lever. Coordination is not the same as direct guidance of the partner's actions, as it often happens, when formal distribution of roles takes place, and the task can be done individually. The evaluation of load with the coins sets the special language for such coordination, and the latter forms the content of the conceptual mediation of their future actions' design. The results of the orientation procedure retrieved from the joint work of the partners, yet has to be reconsidered by each student individually—so that they can decide on their own operations, which have to be performed with weight units within available functions. The distribution of operations with the lever enhances the necessity of the orientation procedure. For many children, joint work with a partner may be the only way to grasp the multiplicative concept.

Our experience in the development of computer support for teaching the balance problem allows us to ponder over the main requirements for software educational design, which has the potential to scaffold students' learning activity (concept acquisition).

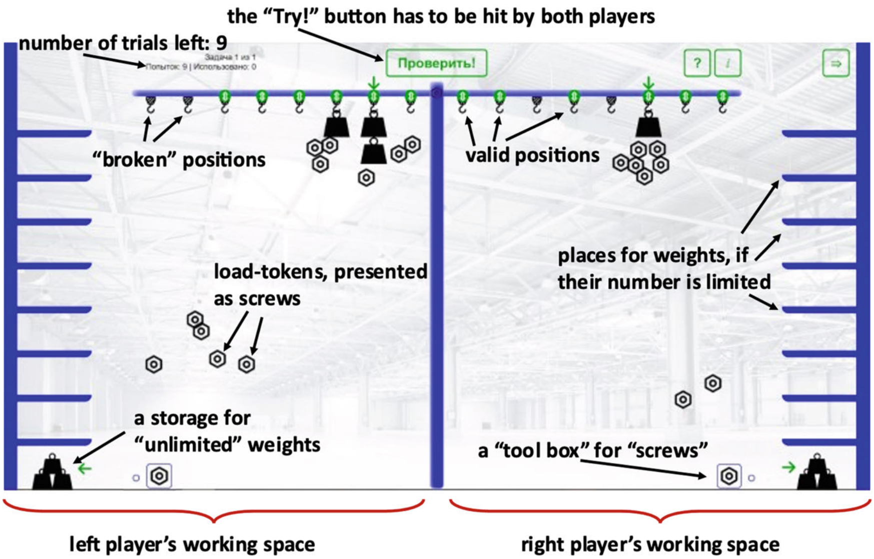

Fig. 10.10 The screen of our digital lever. A "workshop episode" of students' joint problem-solving: each student can rearrange the weight units on his side of the lever. "Screws" are a symbolic means to mark the load for each weight unit. The student on the right has already evaluated the load of his single weight unit, the student on the left tries to balance it with three weight units and is in process of load-evaluation before the balance will be checked

10.4 Discussion

The design of the modelling space, which will be provided for students to make their "thinking" tangible, is the central challenge for the learning activity approach. The special modelling means are introduced in order to scaffold new orientative actions. As students test and adopt them, they rebuild the "natural" image of their usual practical operations with the matter and cease to make well-known mistakes, commonly interpreted as age-related. Our main focus, thus, is on the way to provide (and for students to acquire) the concepts, which prevent students from these mistakes in principle, rather than to investigate "natural" casually happening evolutions of students' ideas and misconceptions. Hence, the appropriate age period for students to acquire the concepts within the learning activity approach depends upon the major task, which the curriculum designers and teachers set before them. For students from 12 years and up the propaedeutics of the natural science concepts is relevant. The formation of functional pre-concepts will scaffold the acquisition and application of the concepts with complex structure (involving the relations between magnitudes, such as a "true" torque concept behind the equilibrium). On the other hand, for primary school students (who we aim mostly at) it is most important to overcome their everyday notions and salient representations by means of special modelling procedures. The "meaningful" rather than "formal" content of these procedures appear to be the gateway to "conceptual" (theoretical) thinking (Davydov, 2008; Davydov & Vardanyan, 1981), which is the central psychological attainment for this age period.

Our experimental teaching series on the balance concept were conducted with students of different age (from 2nd grade students to adults) and proved the very possibility to set before students the learning task as planned. For 6th and 7th graders, it was a relevant educational module in line with the general topic of simple machines, which is a starting one for learning Physics. Concerning the primary school students, we were not aiming to provide them with a grown-up concept of the "torque", we rather gave them opportunity to act conceptually when solving a comprehensible, yet challenging problem in a way, which contradicts their naïve suggestions based on consideration of one parameter, and on additive strategies, they adhere.

The distribution of object-oriented actions between partners, which we exploit, parts its ways from the trending comprehension of collaboration and cooperation in learning (Sun et al., 2020). A number of difficulties in problem-solving based on the necessity to consider the changes of two parameters simultaneously, may be overcome through the organisation of joint actions. "Keeping in mind" the inner coordination of the two factor's changes was no more a problem, as this relation was "kept" by external means, which scaffolded the distribution and coordination of the joint changes performed on the lever. The problem of bringing together the disparate heterogeneous factors was resolved on its own – it was replaced by the other task: students were to choose the operations and correct their ways of changing the object (lever), adjusting to the desired result (balance) defined by general means (load-tokens) beforehand. This path of concept formation allows students to consider right from the start a variety of problems and tasks, much broader than it was thought to be possible within the analysis of "two factor's interrelations". Our tasks included "scattered" weights, balancing unknown weights, etc., which have always been difficult even for adults, not to mention primary school graduates and younger.

10.5 Conclusion

The digitalisation of learning urges the educational designers once again to question the "activity" content of school subjects and raises expectations and requirements towards the logical and psychological analysis, which yet has to be conducted for each concept students need to acquire. The potential of computer support depends on the structure and content of the learning activity, which is specially designed for a particular school concept formation. Unless the content of the concept of interest is analysed in terms of special orientation procedures, unless the situation of concept origin is reconstructed within students' own learning activity, the role and functions of a computer support cannot be determined. As we reconsider the most researched upon task of balancing a lever within the learning activity approach, we can make a step away from the traditional comprehension of balance as the multiplication of "two factors" changing independently, and the corresponding educational design. The consideration of weight and distance as such appears to be insufficient to form the wholesome orientation base for students to solve all tasks of the kind. On the contrary, the well-known inverse proportion may be understood itself as a reflection and a derivate of already-acquired concept in a particular case of balance (that is, when weights are not "scattered").

It is remarkable, that the approach to educational design, based on the cultural-historical and activity theory, helps to determine computer's role in scaffolding students' own learning activity within the substantial communication. Computer support proves to be essential for delivering the balance concept: it helps to set the problem, it enables modelling with special objects directly on the screen, it allows to create tasks with a variety of modifications to constantly return students to the conceptual grounds of their actions in the appropriate orientation procedures. The most essential function of computer software is supporting joint actions: the procedure and the means of coordination of partners' partial actions comprise the content of their future concept.

As researchers choose to deal directly with some aspects of computer-based education, they leave the wholesome picture of concept acquisition out of scope – consequently, for task design they exploit traditional instruction or try to avoid any subject-specific content at all, suggesting some logical and "real-life" tasks instead (Alterman & Harsch, 2017; Care et al., 2016; Goodyear et al., 2004; Schellens & Valcke, 2005). As students' actions and actual orientation processes are adopted from traditional didactics, the outcome in terms of subjects' comprehension does not change drastically (Lowe, 2001), though some gains may be observed in communicative competencies or motivation.

Computer support design, based on the learning activity approach, aims to set a learning problem (in Davydov's classical definition, as a step to the search for general method for solving particular tasks). The appropriate way to build the "conceptual" method of problem–solving may be demonstrated on the multiplicative concepts example and confirms the basic principles of computer support development, which we have outlined earlier (Rubtsov et al., 2021; Vysotskaya, 1996). The computer support has to include:

- conducting "one-side" transformations of the objects, which will reveal the necessity of the conceptual connection between the "distributed" actions;
- testing the new means of coordination within the modelling space, adjusting the models of objects to the verified models of actions;
- the retention of the way, in which the parameters of the objects are interrelated, with the new notion, which is used to "separate" the actions from the particular situation;
- joint design of the new schemes of operations to transform objects according to the tasks' requirements.

The described feasible way to support students' dealing with the troublesome "multiplicative" tasks (in the example of the balance problems) through the specially designed computer simulation may be considered original among a great number of approaches to digitalisation in education. It may be sufficient for integration of communicative and conceptual components of the task based on the meaningful joint model-mediated actions.

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Chapter 11
The Math Club as a Learning Space for Teachers: Guiding Principles Based on Cultural-Historical Theory

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Abstract Considering that the teacher education must offer opportunities for student teachers to be perceived as a regular teacher, in a process of practices supported by reflection. The Mathematics Club project has involved students from mathematics and pedagogy, which consists of a learning space for teaching education once it offers the student teachers a chance to organize, to apply as well as to plan teaching activities. Above all, the student teachers involved in the project ought to share their knowledge and experiences in order to reflect on the pedagogical action. The teacher's work is based on the idea that one cannot enclose the practice through individualities due to it is something necessarily shared; therefore, all actions are carried out by everybody involved in the project. The Mathematics Club is developed in a perspective of cultural-historical activity theory. In this way, the student teacher searches for conditions that involve him to organize his teaching in order to transform it into his own learning.

When we examine the teacher's activity, we see that its significance is found in the act of teaching. In certain ways, the teacher's job has intrinsic autonomy, yet this autonomy is insufficient to prevent alienation. The subjective nature of this activity, in our opinion, is the primary cause of this alienation. The teacher education is one of the subjective conditions of instructional activity. This training process is unable to offer the individual with a grasp of the educational activity's purpose. Simply said,

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it reduces the teacher to little more than a practitioner or someone who engages in contemplation, but this in turn is completely devoid of significance.

By defining teaching as the fundamental element of educational activity, we recognize that this demands an effort to comprehend not just the teaching and learning processes, but also the concerns linked to the spaces where these processes unfold.

In this way, teacher preparation assumes that they will develop professionally while completely avoiding the pitfalls of professionalism. In the meanwhile, it links them to the appropriation of the teaching activity's meaning, which enables the complete and unhindered improvement of human condition. From this assertion, we can infer the following about what a math teacher is:

> [...] the subject who dominates the content, but more importantly, the one who has a strategic vision of their action in the school teaching project in which mathematics has a certain cultural and formative value and, thus, by executing it, they build themselves up with new qualities as teachers while also building with students a mathematics to humanize their worlds.[1] (Moura, 2000, p. 126)

To reach this profile of a mathematics teacher, it is important to realize that not all types of training enable instructors to acquire the abilities and knowledge inherent in pedagogical activity, i.e., the school does not always assist to think better. In this regard, we should consider the following:

> The priority for those who disagree with the way society [*in our case, school, and teacher education*] is to work together to create effective places for innovation in educational practice that each individual produces in their own institution.[2] (Cortella, 2000, p. 137)

Recognizing the limitations in the overall training of a teacher who teach mathematics, and having dissatisfaction with the way school learning is organized as a motive, the Math Club project was created, which answers the demand for the creation of effective spaces for innovation (Cortella, 2000), or in our terms, learning spaces.

It should be noted that the Math Club is not comprehended in its typical format (Graven, 2015; Stott & Graven, 2013). The formative suggestion of the Mathematical Club (CluMat) differs in this manner from the way we often implement formative processes for instructors of mathematics since most of the activities are organized collaboratively, and this is done in consideration of the crucial articulation between theory and practice, which is one of the characteristics of the Math Club's educational proposal. As a result, the subjects who take part in it might assign alternative meanings to the organizing acts of mathematical education.

[1] "[...] o sujeito que domina o conteúdo, mas é, sobretudo, o que tem a visão estratégica da sua ação no projeto de ensino da escola em que a Matemática tem um determinado valor cultural e formativo e, sendo assim, ao executá-lo, edifica-se com novas qualidades de professor ao mesmo tempo em que constrói com os alunos uma Matemática humanizadora de seus mundos".

[2] "O prioritário, para aqueles que discordam da forma como a sociedade [no nosso caso mais especifico a escola e a formação de professores] se organiza, é construir coletivamente os espaços efetivos de inovação na prática educativa que cada um desenvolve na sua própria instituição".

Studies conducted by Cedro (2004), Lopes (2004), and Moura (2000), among others, demonstrate that in contrast to the theoretical discourse on the typical challenges faced by math instructors, the demands set out by CluMat enable the participant teachers in training to comprehend the impact of such conditions in the planning and development of the educational activity. This generates fresh knowledge as teachers in training realize how important it is to engage in spaces marked by collaboration.

11.1 The Math Club

The Math Club was created as an internship project for pedagogy and mathematics undergraduate students, with the assumption that the teacher's formative process is important in making educational activity more objective (Moura, 2021).

This activity should be regarded as structured in the dimension of human activity (Leontiev, 1978), because in activity, the educator acts from an ideal plan by organizing activities and operations that are carried out for appropriating scientific expertise and knowledge.

The development of proposals for teacher education that reflect the importance of interactions in the process of significance of what they aim at (the formation of theoretical thinking through the appropriation of scientific knowledge) was therefore fundamentally influenced by the theoretical foundations that can open the way for teaching while considering students and teachers from the perspective of subjects in activity (Leontiev, 1978).

In creating venues for the Math Club, we intended to experience ideas produced within paradigms for the conceptualization of schooling at that moment, which included: the teacher's personal dimension; the role of practice in training processes, and the protection of schools as places for teachers' professional development. We primarily intended to create conditions that would allow teachers to practice their profession as subjects who interact with concepts as symbolic tools in their preponderant activity: the pedagogical. Thus, in teacher education centers, these should precisely experience the process of meaning-making of their future activities toward the understanding of their object to bring social significance to what they do.

The Math Club arose naturally from a model of learning processes that was extended to the style of collaborative teacher education, which we later referred to as a training space (Cedro, 2015). Nevertheless, there is a novel quality in this Math Club as a space for learning how to teach: the establishment within the teacher's training unit of a place of accomplishment of pedagogical activity like the one to be concretely experienced when completing the program.

Based on what we believed to be having an impact on the formative processes of the teachers, both those who were already working professionally and those who were in the process of training, the theoretical foundations of the teacher education action that took place in the Math Club were also being shaped and deepened during the activities performed. The execution of the activities is allowed for the awareness-raising of the teacher's formative teaching processes, which were settled on the cultural-historical theory's principles and notably on the concept of activity (Leontiev, 1978).

11.2 Guiding Principles for the Mathematics Teachers Education

We contend that teaching is critical to individual growth and that the relationship between the movement of teaching (the teacher) and learning (the student) is tied to how the educator arranges teaching. "Properly planned learning turns into mental development and sets in action numerous developmental processes that would otherwise be unable to occur," writes Vygotsky (1984, p. 118). As a result, we can deduce the significance of teacher education for a teaching organization focused on student learning.

This idea implies that we need to enhance teacher preparation in order to improve education, a sentiment that most individuals involved in education may be said to share to some level. This notion has caused many agencies and managers to believe that every course, program, project, or activity offered or pushed on teachers would result in enhanced learning and that if this does not occur, teachers are to be blamed. However, this premise is false, because it is critical to grasp that no procedure involving instructors (or prospective teachers) can be deemed formative.

Our experience as a teacher educator, combined with the research findings of our group and others in which we are involved, has led us to consider what elements or aspects—which we refer to as guiding principles at this point—can be considered relevant when discussing teacher education, especially for those who teach mathematics.

Among these, we highlight five:

1. The school as an organized space for appropriation of human culture.
2. The teacher as the actor in training processes.
3. Intentionality as an essential component of teaching organization.
4. Sharing as a basic principle for understanding the pedagogical activity.
5. The role of mathematical knowledge in fostering personal growth.

By writing option, although we will explore each of them separately in this text, they constitute a unit of the object described here—formation—as far as they complete and interrelate at the many times that entail a formative process. In this way, we clarify that we see them as both a premise and a product of any process aimed at becoming formative.

1. *The School as an Organized Space for Appropriation of Human Culture*

In the development of formative processes, it is important to discuss the social role of the school and its commitment the students that it educates.

We will start by turning to Leontiev (1978), who emphasizes that human beings do not naturally possess the historical knowledge that humanity has amassed; rather, this knowledge is developed over time and is absorbed in the world at large, not individually, into the major works of human civilization. Only through appropriation of the great works of human culture during one's life, can one get fully human qualities and faculties. As the author puts it:

> Therefore, each generation begins its life in a world of objects and phenomena created by the previous generations. They appropriate the richness of this world, taking part in the work, the production and
>
> in the varied forms of social activities, thus developing specifically human skills […].[3] (Leontiev, 1978, p. 284)

In this regard, Rigon et al., (2010, p. 29) emphasize that social practice serves as a bridge between the individual and genericity and that "the cultural appropriations inherent in this process coincide with the socialization of historically produced knowledge." The subject's psychic abilities are formed because of their activity of appropriating culture.

> In carrying out tasks, the unique individual interacts with humanity in a mediated manner. The basic interaction that the singular person creates with society is this mediation between the individual and the generic. The individual becomes human throughout life in society by appropriating the human essence, which is a cultural-historical product, through this process of appropriations and objectivations made possible by labor.[4] (Rigon et al., 2010, p. 16)

Humans actively participate in this process, which is not innate since they actively reproduce the characteristics of earlier generations and can use these traits to advance in their own development. This leads us to infer that education facilitates the acquisition of knowledge, techniques, values, practices, behaviors, and art.

From this perspective, education is a humanization process to the extent that knowledge appropriation is seen as a means of gaining access to historically produced culture. We draw attention to the fact that our understanding endorses Martins (2010, p. 16) assertion that

> As a result, we are not referring to a liberal conception of humanization, in which this process is exemplified by the centrality of the abstract subject in relation to the concrete circumstances of its existence. On the contrary, it is a process that depends on the production and reproduction in each unique individual of the highest capacities already attained by the human species. A process, in other words, that is completely constrained by the appropriations of physical and symbolic property produced historically by human labor, from which professors cannot be alienated.[5]

[3] "Cada geração começa, portanto, a sua vida num mundo de objetos e de fenômenos criados pelas gerações precedentes. Ela apropria-se das riquezas desse mundo participando no trabalho, na produção e nas diversas formas de atividade social e desenvolvendo assim as aptidões especificamente humanas".

[4] "Na realização de sua atividade, o homem singular relaciona-se, de forma também mediada, com o gênero humano. Essa mediação entre o indivíduo e a genericidade é a própria relação que o homem singular estabelece com a sociedade. Nesse processo de apropriações e objetivações, viabilizado por meio do trabalho, o indivíduo torna-se humano ao longo de sua vida em sociedade, ao apropriar-se da essência humana, que é um produto histórico-cultural."

[5] "Não estamos, portanto, nos referindo à concepção liberal de humanização, para quem esse processo se efetiva na centralidade do sujeito abstraído das circunstâncias concretas de sua

In this light, we may conclude that the school is a space that facilitates the absorption of historically generated culture, therefore boosting the development of students' maximal potential, particularly through curricular knowledge.

However, in order to develop humanizing abilities, "it is necessary to develop in relation to the object or phenomenon a (adequate) activity that reproduces, by form, the essential features of the embodied activity, accumulated in the object" (Leontiev, 1978, p. 286). It is in this context that the teacher is placed in their teaching activity, because it is expected that at school, the process of cultural appropriation occurs, and the teacher, while not.

Moura (2001) clarifies that the student and teacher to engage in humanizing activity, they must "imagine" the object that satisfies their needs to humanize themselves. This object in the educational system is knowledge, and in the case of teachers, this object is the activity that these actions are composed of:

> Being a teacher means that we have a common characteristic with other people who have as their main practice to teach something to someone else, that is, to be a teacher requires an action that aims to transform oneself by transforming another person, to change their way of being and acting. We believe that the subject, who is the fruit of our educational action, will acquire a certain knowledge that will enable them to act in a certain way in the environment in which they live. Their own learning will enable them to understand some phenomenon in some way. And this will allow for using this new knowledge to impact reality.[6] (Moura, 2001, p. 144)

A formative approach must ensure that the teacher understands that the actions they organize can improve the development of their students' education. Therefore, it is important to learn how to encourage educational actions that fully enable the development of human qualities possible, whether by choosing materials that have to be adapted or by setting up learning trigger situations that push students to look for solutions to a problem that arises from the need to appropriate the knowledge required for this. In other words, the teaching organization should lead to the absorption of knowledge that humanity has historically accumulated.

Based on the foundations that inspire us, we repeat that education is a vital component of human development. Paro (2001) contributes to this concept by stating that although completely devoid of cultural qualities at birth, people are the authors of history when they produce culture based on their socially constructed needs and that education enables the appropriation of this human history. We argue that individuals are provided education as a cultural-historical update, which means that.

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existência. Trata-se, outrossim, de um processo dependente da produção e reprodução em cada indivíduo particular das máximas capacidades já conquistadas pelo gênero humano. Um processo, portanto, absolutamente condicionado pelas apropriações do patrimônio físico simbólico produzido historicamente pelo trabalho dos homens, dos quais os professores não podem estar alienados."

[6] "O fato de ser professor diz que temos uma característica comum com outros sujeitos que têm como prática principal ensinar algo a alguém, isto é, para ser professor é necessária uma ação que visa transformar-se ao transformar outra pessoa, mudar o seu modo de ser e de agir. Acreditamos que o sujeito, que é fruto de nossa ação educativa, vai adquirir um certo conhecimento que vai lhe capacitar a agir de uma determinada forma no meio em que vive. A sua aprendizagem vai lhe capacitar a compreender algum fenômeno de alguma forma. E isto vai lhe permitir usar desse novo saber para impactar a realidade."

> through the appropriation of knowledge, values, beliefs, artistic abilities, etc., people create potentialities that add culture to their nature, making them more human (historically speaking). One becomes a human-historical being through the appropriation of cultural components, which combine to form a living personality.[7] (Paro, 2001, p. 25)

Education, particularly school education, becomes important as a means of human development by allowing a conscious organization of the processes of individual formation from an intentional organization of teaching, providing subjects with the opportunity to appropriate knowledge, skills, and forms of behavior produced by humanity. In this light, "the school is a privileged institution in terms of humanization prospects" (Rigon et al., 2010, p. 29), and the teacher is the subject who must possess the access to the realm of human heritage to pass it on to future generations.

This realm encompasses numerous factors, including knowledge of what will be taught, in the case of the topic at hand, mathematical knowledge, and its role in the humanization process.

2. *The Teacher as the Actor in Training Processes*

We have observed that many courses, programs, projects, or other initiatives were masterfully planned by outstanding professionals in the field of education failed to succeed when they were imposed on instructors. Several studies, including those by Marco et al. (2015) and Silva and Coco (2017), have highlighted the importance of developing training processes that consider the teacher in their professional situation, and, above all, it should go in accordance with their very training needs rather than with others pushed by the people who propose them.

We underline that we agree with Martins (2010, p. 14) that professional training, particularly teacher education, is "a trajectory of training of individuals, intentionally planned, for the realization of a certain social practice." No formative process can, therefore, be separated from the intricate social web that the teaching profession is a part of.

When we refer to the teachers' labor, we assume that their act of teaching, which is directed toward a specific goal (to teach), needs to be planned out accordingly. That is, we understand from a cultural-historical perspective that if the human being is formed through working—defined as a human activity—teachers are formed by aligning their needs with the learning of their students.

Consequently, if the teacher's work is the teaching activity that is related to satisfying the need to teach, the formative processes are likely to be related to their need to provide themselves with knowledge to this end. From this reasoning, we can anticipate a training process to be a training activity.

It should be kept in mind that every activity belongs to an agent; hence, the teacher must be the agent of the training activity. Teachers will only participate in a training activity, though, if it is driven by their own training needs (and not external to them), as a reflection of their motives.

[7] "ele vai se tornando mais humano (histórico) à medida que desenvolve suas potencialidades, que à sua natureza vai acrescentando cultura, pela apropriação de conhecimentos, valores, crenças, habilidades artísticas etc. É pela apropriação dos elementos culturais, que passam a constituir sua personalidade viva, que o homem se faz humano-histórico".

Let us recapitulate Leontiev's (1978, p. 96) notion that motive "does not indicate the sense of a need, but the results of a need that is fulfilled with objectives in the conditions presented and toward which the activity is oriented, what motivates it." And the purpose that drives the subject's actions is what generates meanings.

But why does a teacher decide to take part in a formative process?

If it has to do with fulfilling an obligation to the school or the department of education, the incentive will only act as a motivator. Given that the motivation does not align with the goal of the training activity, the teacher may even attend the entire process and perform the activities just with the objective of finishing them.

On the other hand, if the training process unfolds from their desires to acquire the necessary knowledge for their jobs, there is a chance that their split will be motivated by the personal meanings associated with their training. Remember that the senses are the individual consciousness' reflection of the appropriation of a given social meaning and, as such, are strongly tied to what is specific to the subject. The motives established in this manner will be effective because they are tied to the meanings assigned to the training process by the teacher (rather than the organizer) and they can induce a training activity.

When we understand that training processes should provide opportunities for teachers to acquire knowledge aimed at developing their teaching activity, we emphasize that they should take an active role in organization and implementation, rather than being mere executors of tasks or formulas.

Franco and Longarezi (2011, p. 564) highlight the importance of

> […] the subject's engagement and involvement in their formative process as directly tied to the realization of their needs rather than those who offer it. This helps us understand why many ideas, particularly for continuing education, frequently fail to achieve their objectives since they are based on external stimuli towards instructors and may lead to alienation, as evidenced by teachers' speeches, behavior, and the school culture.[8]

The writers emphasize the alienation caused by a discordance between the objective consequence of human activity and its motivation, or, to put it another way, when "the objective content of the activity does not currently coincide with its subjective content, that is, with what it is for the very person" (Leontiev, 1978, p. 122). And this can only be prevented if formative processes are organized as formative action with the instructor as the subject.

Furthermore, one must consider their workplace—the school—and its involvement in the education of the people involved.

3. *Intentionality as an Essential Component of Teaching Organization*

So far, we have discussed the possibilities of training processes being organized as a training activity from the perspective of humanization, arguing that knowledge

[8] "[...] a participação e o envolvimento do sujeito em seu processo formativo estejam estreitamente relacionados à satisfação de sua necessidade e não a de quem o propõe. Isso nos leva a compreender porque muitas propostas, principalmente de formação continuada, muitas vezes não atingem seus objetivos por serem propostos a partir de estímulos externos aos professores podendo levar à alienação, o que é corriqueiramente percebido nas falas dos professores, seus comportamentos e na cultura escolar".

historically produced by people is part of human culture and an element of human development, with the social role of the teacher being to organize the teaching in such a way that students can appropriate the developed culture. Additionally, we have previously emphasized the Vygotskian premise that learning and development are not the results of just any education, which is consistent with this view.

To accomplish this, teaching actions must be directed toward attaining the social objectives of the school curriculum, which requires that the teacher define actions, select instruments, and evaluate the teaching and learning process. This is because students can appropriate the most diverse elements of human culture according to their needs and interests in an unintentional and non-systematic manner; however, it is in the intentionally organized process of school education that the appropriation of theoretical knowledge, through which the formation of theoretical thinking that enables psychic development is structured (Moura et al., 2010).

We then assume, aligned with Moura et al., (2017, p. 72), that

> intentionality on the part of the teacher regarding the goal of their activity (teaching), combined with the actions and operations to ensure that a concept is learned, triggers the processes of reflection, analysis, and synthesis by the teacher when interacting with students, which may add new quality to their general method of structuring their Pedagogical Activity.[9]

Therefore, teaching, when intentionally organized, can promote and trigger development processes, because, as Saviani (1991, p. 17) puts it, "educational work is the act of producing, directly and intentionally, in each singular individual, the humanity that is historically and collectively produced by the group of people."

The commitment of those who teach mathematics to their students' learning is represented in the intentionality of their activities, which attempt to transform education into a learning experience for the student, with theoretical knowledge serving as a reference in the humanization process.

> The teacher's instructional activity must generate and promote student's activity. It should instill a unique motivation for them to study and acquire theoretical knowledge of reality. Teachers arrange their personal activities, as well as their orientation, organization, and evaluation actions, with this goal in mind.[10] (Moura et al., 2010, p. 90)

We should not forget that Leontiev (1978) claimed that human activity is defined by its purposeful nature since not only all acts are oriented toward a goal and motivated by the satisfaction of needs, but are also grounded on the motives, and that this movement results in the development of better cognitive functions.

As a result, the teacher's activity, which involves the organization of teaching aimed at student learning, implies that teachers are aware of their actions that will

[9] "a intencionalidade do professor acerca da objetivação de sua atividade - o ensino -, aliada às ações e operações para propiciar a aprendizagem de um conceito, desencadeia os processos de reflexão, análise e síntese por parte do professor ao interagir com os estudantes, o que poderá dar nova qualidade ao seu modo geral de organizar sua Atividade Pedagógica".

[10] "A atividade de ensino do professor deve gerar e promover a atividade do estudante. Ela deve criar nele um motivo especial para a sua atividade: estudar e aprender teoricamente sobre a realidade. É com essa intenção que o professor planeja a sua própria atividade e suas ações de orientação, organização e avaliação".

intentionality guide it, determining how to overcome the unconscious and arbitrary plane. So, being engaged in a teaching activity means being aware of the act of teaching.

> More than anything else, consciousness entails being aware of the association to the community whose actions are intended to provide the appropriation of human culture, or more precisely, the appropriation of symbolic objects that can give the subjects the means necessary to actively participate in society.[11] (Moura, 2001, p. 186)

Because it is the teacher's responsibility to plan actions that, when implemented, will allow the student to acquire knowledge, as we have often maintained, intentionality is an essential component in organizing teaching.

The development of the student's higher psychological functions becomes dependent upon intentionally planned instruction. But we want to be clear that the teacher, who is the agent of the teaching action, is the one who must be intentional. Likewise, this (intentionality) places accountability on those involved in lesson planning, which warrants its treatment as a crucial factor to take into account in the training process.

Nonetheless, it should also be noted that no process becomes formative when it focuses exclusively on the individuality of the subjects that compose it.

4. *Sharing as a Basic Principle for Understanding the Pedagogical Activity*

When Vygotsky claims that a subject learns through interaction with others, he is referring to the subjects' active participation. This premise aids in our comprehension of how sharing can advance the learning of teaching.

By collaboratively designing learning actions, teachers create a new concept of pedagogical activity, allowing them to appropriate the meaning of this activity, which should be object-oriented (teaching); driven by needs (to help the subject learn); and motivated by motives (to bring the subject closer to the knowledge produced by humanity).

Sharing actions, perceptions, and meanings implies interaction between multiple individuals, each with their own set of knowledge, which might influence the quality of the process in which they are involved. This is because collective reflections require that the subjects must be guided by common objectives to achieve new learning, as long as

> education is a collective process, [for] it is in sharing that the teacher can appropriate new knowledge, because, although the actions may be of each of those who perform a particular activity, learning does not happen in what each one does in isolation, but in the interaction between subjects or between subjects and objects. Thus, it is necessary that the actions are developed by all, but that each one has not only the opportunity, but also the commitment to participate[12] (Lopes et al., 2016, p. 25).

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[11] "Consciência que é acima de tudo, a de ser pertencente a uma comunidade cuja ação tem por finalidade propiciar a apropriação da cultura humana, ou mais objetivamente, a apropriação de ferramentas simbólicas capazes de permitir aos sujeitos os meios necessários para viverem plenamente em sociedade".

[12] "sendo a educação um processo coletivo, é no compartilhar que o docente tem a oportunidade de apropriar-se de novos conhecimentos, pois, embora as ações possam ser de cada um daqueles que

We argue that it is critical in the process of teacher education to create situations in which there is a need to share actions, because doing so allows subjects to develop specific forms of cooperation, which may allow them to reach an appropriate level in cognitive actions through appropriation and awareness of the significant process of collective knowledge production.

Therefore, the teaching-learning process, as defined as an activity—like we previously established—is made possible through interactions directed by similar goals. This allows it to manifest in the activities carried out with the other subjects. In this process of interactions, individual actions are coordinated by identifying an object's properties, changing them, and creating collective effects. As a result, collaborative actions that allow information exchange make it possible for subjects to reflect on their own learning while considering the contributions of others.

"The sharing of actions manifests itself in a fruitful cognitive activity," according to Polivanova (1996, p. 151), "via a high level of structuring of intellectual effort, and in an intensified form of reflection, control, and evaluation." Thus, learning to teach, which involves sharing actions and knowledge with other subjects who have similar needs, enables teachers to create new connections that result in the acquisition of new knowledge.

It should be highlighted that our definition of shared training extends beyond performing actions together. It entails carrying out collaborative and interactive actions capable of fostering the growth of the subjects involved, as well as developing common objectives aimed at ways of organizing education geared toward students and satisfying both individual and collective needs.

In this way, we can understand that, in a formative process, while what stimulates each subject's participation may initially respond to individual needs, it is the sharing of actions that will ultimately determine the results obtained, so that individual actions do not represent the product of joint activity, which can only be obtained through collective actions originating in social relationships.

This perspective returns to the Vygotskyian principle, already mentioned here, that the general trend of psychic development always goes from the social to the individual, which leads us to infer that an action shared among several subjects can become a general mode of organization of each of them, that is, the interpsychic action becomes intrapsychic. This is how teaching training through sharing occurs: when common actions among teachers aimed at organizing teaching become general modes of organizing everyone's teaching.

Furthermore, the subject might discuss their own activity based on what other subjects say in the interaction, or the collective look for solutions to a particular situation can become the individual solution. It is the collective view on the totality of pedagogical activity—which includes both teaching and learning—that allows teachers to appropriate the meaning of this activity, which must: be focused on an

concretizam uma determinada atividade, a aprendizagem não acontece no que cada um deles faz de forma isolada, mas na interação entre sujeitos ou entre sujeitos e objetos. Assim, faz-se necessário que as ações sejam desenvolvidas por todos, mas que cada um tenha não só a oportunidade, mas o comprometimento de participar".

object—teaching; be driven by needs—to make the subject learn; and have motives—to bring the subject closer to the body of knowledge that humankind has explored.

Therefore, formative processes that take as fundamental the sharing among different individuals, who occupy different spaces in education, open up possibilities for better discussion about teaching and learning and all the issues that influence them, allowing all those involved in the pedagogical activity to understand themselves as a unit between teaching and learning activities, organizing themselves in a teacher learning space.

5. *The Role of Mathematical Knowledge in Fostering Personal Growth*

We believe that the school promotes physical, motor, psychological, and emotional growth, as well as the acquisition and development of skills, values, attitudes, behaviors, and relationship patterns via experiences. But we draw special emphasis to the school's dedication to the development of theoretical thinking, via theoretical (scientific) knowledge historically created and recognized as a fundamental prerequisite for the development of subjects (Davidov, 1988), both those who learn and those who teach.

Moura (2001) explains that being a mathematics teacher encompasses two dualities to be considered.

> The first, that of being a teacher, describes teaching as a profession that deals with a specific understanding, deemed as pertinent to be taught at school, of how to organize socially produced education. The second is the teacher's level of specialization. In this example, we took a math teacher. This denotes a completely different quality than, say, a teacher of physics.[13] (Moura, 2001, pp. 147–148)

The author explains that the method of teaching is related to the nature of knowledge and that the contents of this teaching can be thought of as syntheses produced by humanity when dealing with problems—the fruits of their needs. And these syntheses were chosen as relevant to be conveyed at a given time to allow the integration of new subjects into the dynamics of the group to which they belong.

We can regard mathematical knowledge as a cultural heritage historically produced by humanity, and, as such, using it promotes the development of higher psychological functions. However, as previously said, we are not referring to any knowledge, but rather to theoretical knowledge.

Our theoretical assumptions explain that there are two types of knowledge (empirical and theoretical) directly associated to two lines of thinking (empirical and theoretical).

As stated by Davidov (1988), empirical concepts do not reveal the object's internal and essential connections as they are based on common attributes, sensory experiences, and are presented in everyday life. Even though empirical generalization is

[13] "A primeira, a de ser professor, identifica-o como uma profissão de quem tem um certo conhecimento sobre o modo de organizar o ensino produzido socialmente e tido coo relevante para ser veiculado na escola. A segunda é a especificidade do professor. Ele, nesse caso, é professor de Matemática. Isso significa uma qualidade muito diferente daquela que é ser um professor de Física, por exemplo."

significant, it is only the first step in the knowledge process because it only identifies common aspects of the object in each specific and concrete instance. These common aspects are primarily the result of observation and perception, most frequently from a comparison of related facts or individual properties. Empirical knowledge, also known as spontaneous or everyday conceptualization, refers to ideas that are fostered by a subject's practical activity and in-person interaction with their environment.

Davidov (1988) also points out that teaching with a predominance of the intuitive approach, in the utilitarian and empirical dimension of knowledge, is vital for everyday activities, during the completion of regular labor actions, but it is insufficient to assimilate the authentic spirit of contemporary science.

In contrast to empirical thinking, the essence of theoretical thinking formation processes "is a specific procedure through which people focus on comprehending things and occurrences through the analysis of their origin and growth" (Davidov, 1988, p. 6). Theory-based thinking, in other words, "constitutes a type of thinking that aims to recreate the essence of the object studied in the course of the formation of mental actions that occurs in the intentional process of teaching for development" (Davidov, 1988, p. 10).

According to Kopnin (1978, p. 153), the theoretical thinking reflects the internal relationships of the object, and its laws of motion are accessible through rational development and a system of data abstractions from empirical knowledge. The distinction between the two is up to a certain point conditional because "the empirical becomes the theoretical and, on the other hand, what at one stage of the development of science was thought to be purely theoretical becomes empirically accessible at a later stage" (Kopnin, 1978, p. 153).

It is worth mentioning that theoretical (or scientific) conceptions as we know them now were historically formed with the involvement of diverse civilizations, addressing diverse needs, in diverse public, economic, political, and cultural situations. Understanding a notion, object, or phenomena requires knowledge of its logical-historical movement, which explains the links between its elements and demonstrates its internal connections.

So, knowing that mathematical ideas were historically shaped by human needs is important because it helps teachers organize their lessons in a logical and historical manner, which in turn helps students comprehend the historically logical path of knowledge creation. The connection between the logical and the historical, as Kopnin (1978) explains it, enables comprehension of the object's essence.

> By historical, we mean the object's changing process, the stages of its emergence and development. The historical acts as an object of thought, a mirror of the historical, and a content. Thought seeks to reproduce the actual historical process in all its objectivity, complexity, and contradiction. The logical is the technique by which mind fulfills this duty, but it reflects the historical in a theoretical form, that is, it is the replication of the essence of the object and its history in the system of abstractions.[14] (Kopnin, 1978, pp. 183–184)

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[14] "Por histórico subentende-se o processo de mudança do objeto, as etapas do seu surgimento e do seu desenvolvimento. O histórico atua como objeto do pensamento, o reflexo do histórico, como conteúdo. O pensamento visa a reprodução do processo histórico real em toda a sua objetividade, complexidade e contrariedade. O lógico é o meio através do qual o pensamento realiza esta tarefa,

From what has been shown, we infer how important it is for methodological proposals to have the history of conceptualization as a guiding factor, embodied in effective learning situations that bring the concept's essence with them. But "the history of mathematics that incorporates situations to sparkle learning is not the factual history, but the history that is impregnated in the concept, given that the concept tries to meet a historically positioned human need" (Moretti, 2007, p. 98). In this approach, the activity of teaching mathematics will "allow the professor and the student to perceive this science as a human product" (Rigon et al., 2010, p. 148) and, from this perspective, the acquisition of knowledge, in this case mathematics, will promote the development of both the student and the teacher.

Overall, considering that the application of theoretical knowledge is one of the key components for the development of the subject's potentiality and that the logical-historical movement should guide the organization of teaching, presupposes the fact that teachers, each of them in a formative process, need to experience opportunities to advance their theoretical thinking through mathematical knowledge. In this way, they can absorb pertinent aspects for the intentional organization of their teaching.

11.3 A Provisional Overview

We give some ideas that have shown to be beneficial in directing our activities in connection to the educational processes in which we are involved, so we can have a discussion on how math teachers are trained. With the intention of bringing about changes in pedagogical action to better benefit students, the discussion was grounded on the notion that not any process that involves teachers qualifies as formative, just the ones that successfully promote knowledge acquisition among teachers.

We sustain that the formative processes will become training activities for teachers solely when they are implemented in accordance to their formative needs, which implies the commitment of training institutions and, mainly, of trainers to collectively devise the courses, programs, projects, or actions for and *along with* teachers. We emphasize how important it is for teachers, during their training activities, to consider the school as the place organized for the appropriation of human culture. This also involves assuming their role in organizing actions that allow their students to appropriate this culture, fostering their development. To that end, it is critical that educational processes consider how the appropriation of theoretical knowledge related to mathematics—which reflects its logical-historical motion—can enable the development of both the teacher and the student. We also draw the conclusion that not just any teaching will result in development and that an organization guided by intentionality, aimed at transforming the teaching activity into a learning activity, is needed.

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mas é o reflexo do histórico em forma teórica, vale dizer, é a reprodução da essência do objeto e da história do seu desenvolvimento no sistema de abstrações".

Finally, we argue that sharing is essential for understanding pedagogical activity because teachers appropriate new knowledge by interacting with other subjects who have the same need of learning how to teach.

We state that the principles described are held as a premise and a product of the formative processes. They are a premise when they guide the actions of the process organization and become a product when they materialize in the teachers' learning.

We also clarify that the principles listed are derived from the theoretical foundations on which we rely, specifically those related to cultural-historical theory. Furthermore, we agree with the ideas of other colleagues who are members of the groups in which we participate, because they help us understand that formative processes should not only lead to the appropriation of knowledge related specifically to the performance of educators, but should also allow them to shape working conditions so that they can establish themselves as agents of their knowledge—built with others.

Lastly, we acknowledge that the discussion of these principles presented here is not over, not only due to the text's limited space, but also the fact that they are not unique, given the complexity of teacher training, in which a variety of relationships and determinations occur. Also, and most importantly, because they refer to formative processes understood as human activities, they are historically and culturally constituted to meet diverse needs that may change, but that, in our perception, should always belong to the subjects who make them up: the teachers.

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Chapter 12
The Organization of Teaching for the Development of Students' Mathematical Conceptual Thinking as Content-form of the Teaching Training Process: A Cultural-Historical Experience in the Context of Brazilian Public Schools

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Abstract The theme encompasses the development of mathematical conceptual thinking from a cultural-historical experience in the context of the Brazilian public school based on the historical-dialectical materialist method, whose theoretical-methodological orientation favors subjects to interrelate their pedagogical and study activities through didactic-training intervention procedure. The purpose of this text is to discuss how the didactic organization aimed at the development of students' mathematical conceptual thinking constitutes a content-form of development of those involved. Results indicate that the class teacher presents the development of mental action on teaching, offering developmental didactic elements capable of guiding students to think and act through the essence of scientific concepts in algebra. Students achieved a certain mastery of the means and of the logical mental actions of the internal movement represented by the theoretical concept of functions and linear and quadratic equations, based on the essence of the links and nexuses that compose them.

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Keywords Mathematical conceptual thinking · Cultural-historical theory · Developmental didactics · Algebra learning

12.1 Introduction

It is based on the premises of the historical-dialectical method and the theoretical-methodological elements of the cultural-historical perspective of activity and developmental didactics (Puentes and Longarezi, 2017), to discuss how the didactic organization focused on the development of students' mathematical conceptual thinking is constituted as a content-form of development of those involved, teachers and students.

The topic has its genesis in a cultural-historical experience in the context of the Brazilian public school of Minas Gerais, whose doctoral study/research was developed with a mathematics teacher and a class of 21 students of basic level II (EF-II), through a didactic and formative intervention, during three semesters (8th and 9th year of basic level II).

In this study/research, we sought to understand the processes of formation and development of teacher and student motives in pedagogical activity, particularly in the formation of certain mathematical concepts in the field of algebra, such as: fractional equations, linear and quadratic equations, and functions. Thus, the study and the organization of the pedagogical activity have been related in a dialectical way, with the purpose of grasping the didactic actions that promote the development of the students and the teacher, in activity conditions. By activity we mean "specific processes that exercise one or another vital, that is, active, relationship between the subject and reality" (Leontiev, 1974, p. 43). The guiding questions of the movement undertaken in the research were: Would it be possible to establish other relationships in the internal structure of pedagogical and study activities with the conditions of Brazilian public education? What didactic and formative actions would be necessary for the constitution of more favorable relations for the development of the motives that form the meaning of these activities?

In this chapter, in particular, we sought to present the understanding of such questions, based on the discussion of the didactic and formative intervention movement developed by the teacher with students in a public school, with the purpose of discussing how the didactic organization focused on the development of the students' mathematical conceptual thinking constitutes content-form of development of those involved, in its relationship with the educational work.

For this purpose, first of all, the concept of activity, in the sense attributed by Leontiev (1978, 2001), is developed as fundamental in the construction of what is proposed. Then, the unfolding of the guiding questions and the new perspectives suggested for their confrontation/surpassing are demonstrated. Then, we discuss the method and the didactic and formative intervention, drivers of the development of the conceptual thinking of the students and the teacher in her pedagogical activity.

Finally, signs of resistance, ruptures, and elements of the subjects' development that indicate the possibilities of this path are demonstrated.

12.2 The Delineations of the Meanings of the Subjects' Activities

Alexis Nikolaevich Leontiev's (Leontiev, 1978, 2001) concept of activity is linked to how the human psyche develops and is determined by man's own real-life process. This means that the place occupied by the subject in the system of social relations depends on the conditions in which he lives and the activity he performs.

Leontiev (1978) argues that some types of activity are more decisive in the development of psychism, and others play a secondary role. Each stage of the development of the psyche is determined by the subject's dominant relationship with reality, with the type of guiding activity peculiar to each period. Guiding activity is understood as that which exerts psychic motor forces and not that in which the subject is involved most of the time. From this concept, it is understood that:

> [...] this activity influences the main changes in psychic processes, so that, at each stage of psychic life, an internal contradiction arises caused by the change of place of the subject in the system of social relations and by the change of the content of the essential activity in this period. In activity there is a triadic relationship (subject-activity-object), constituting historically in life in society, in the processes of collective work and human needs oriented by purposes (Franco et al., 2019, p.707).

Recent scientific studies (Franco, 2009, 2021; Franco & Longarezi, 2011; Franco & Sousa, 2018; Franco et al., 2015, 2016, 2017, 2018; Longarezi & Franco, 2013, 2017; Rodrigues et al., 2018; Longarezi, 2019) have revealed the split between the social and personal meanings of teacher training actions from the perspective of the subjects of the process themselves. These studies, guided by a cultural-historical approach to the activity (Puentes; Longarezi, 2017), show that the reasons for the schism are usually due to the non-satisfaction of the formative needs in the object of the action, because the content and the forms (the mediating instruments) in many of these formative actions do not correspond. According to Franco, Longarezi, and Marco (2019a, 2019b, p. 708), in such conditions, meanings and meanings become contradictory.

Research (Franco, 2009, 2021; Franco & Longarezi, 2011; Franco & Sousa, 2018; Franco et al., 2015, 2016, 2017, 2018; Longarezi & Franco, 2013, 2017; Rodrigues et al., 2018) shows the close relationship between motives, needs, and objective and subjective conditions of pedagogical activity that does not correspond, with a strong impact on the composition of teachers' motives (Franco et al., 2019). Therefore, the Brazilian reality is paradoxical: In many cases, municipal and state secretariats of education offer training actions to teachers in the same proportion as their dissatisfaction with the conditions and contents of what is offered is accentuated.

In this context, how can we expect teachers to develop pedagogical practice that allows students to develop concepts in the different areas of knowledge if they themselves have not been conceptually trained? Guided by this question, we argue that the process of educating teachers and students must be situated in the terrain of human development in social practice, and as such, work (teaching) and education (development) must be considered as a dialectical unit. In this case, pedagogical and study activities are considered as a unit.

Thus, it is understood that in the didactic and formative intervention procedure, the teacher organizes his activity having in mind the students' study activity on the object of scientific knowledge, and as a result, the appropriation of the theoretical concept can be produced. The content and form of this appropriation must be taken as an object of analysis by teachers, because knowing how to guide the student to think and act theoretically goes far beyond the transmission of school content.

According to (Petrovsky 1986, p. 292—our own translation), "[…] thinking is an active process of the subject about the objective world in concepts, judgments, theories, etc.". Therefore, guiding the student in this process is fundamental in school. Furthermore, "[…] to have a concept about one or another object means to know how to mentally reproduce its content, that is, to mentally construct it. The act of constructing and transforming the mental object constitutes the act of understanding and explaining it, of discovering its essence" (Davydov, 1988, p. 126—our translation). For this reason, school education should be concerned not only with knowing whether or not students have mastered the school content they are taught, but also with knowing whether students know how to think conceptually and how education can contribute to this process (Franco et al., 2019).

The aforementioned authors reiterate that the problem presented in this way strengthens the relationship between process and product and the purposes of school education, showing that there is no separation between objective-means and objective-end, because the objectives in school education must be interwoven. Therefore, they can be considered as a theoretical-practical and content-form unity in the education of teachers and students.

The theoretical-methodological approach advocated in this chapter under the name of Obutchénie por Unidades (see more in: Longarezi, 2021a, 2021b; Longarezi & Dias De Sousa, 2019; Franco et al., 2019), from which the confrontation of the issues of school education in Brazil is proposed, in relation to the processes of humanization, in view of the social practice in a concrete context. At the same time, such an approach appears to be a great challenge for all involved, given the complexity and depth of this type of teacher and student training, since it implies facing contradictions, provoking confrontations, dealing with resistance and clashes inherent to the tensions that arise from the different subjects in the different contexts in which they are inserted.

12.3 The Research, Pedagogical, and Study Activity Unit in the Didactic-Formative Procedure: the Investigative Methodology

The didactic-formative intervention was developed in Brazil by researchers from the Group of Studies and Research in Developmental Didactics and Teacher Professionalization (Gepedi), of the Federal University of Uberlândia (UFU) (Ferreira, 2017a, 2017b, 2021, 2018, Franco, 2015; Franco et al., 2018; Germanos, 2016; Longarezi, 2012, 2014, 2017; Longarezi & Ferreira, 2019; Longarezi & Moura, 2017; Sousa, 2016; Marra, 2018; Coelho, 2020; Jesus, 2022), to carry out investigations with developmental didactics (Coelho, 2020; Ferola, 2016; Ferreira, 2017c, 2021; Franco, 2015; Germanos, 2016; Jesus, 2022; Marra, 2018; Sousa, 2016; Souza, 2016), based on historical-dialectical materialism and cultural-historical activity psychology (Puentes and Longarezi, 2017). This type of didactic and formative research and intervention implies active and creative actions of teachers, students, and researchers in their social practices. In its essence, this procedure presents "[…] as a final goal the development of students through the study activity (objective-mean) […]" (Longarezi, 2014, p. 15). It is organized in this sense:

> [...] in four stages: (1) diagnosis of the education-learning-development processes in force in the school contexts; (2) formative actions with the teachers; (3) educational actions with the students; and (4) systematization and analysis of the guiding didactic principles of a learning that triggers development [...]. (Longarezi, 2012, p. 19; our changes).

The research was developed in the school context and takes place in the process of organizing pedagogical activity, and this, in turn, in the process of organizing the students' study. Thus, the movement of didactic and formative intervention is constituted.

In Franco's research (2015), an effective mathematics teacher from the municipal public school manifests the need to identify the students' reasons for studying or not studying mathematics, and thus, to analyze her own pedagogical practice. Based on the assumption that the need, in the sense attributed by Leontiev (1978), should be of the subject and be oriented to solve some theoretical problem of his social practice—in this case, the pedagogical activity, nothing more coherent than to infer from the subject himself his formative need; this was, therefore, the main reason that led to the selection of this teacher, in the development of Franco's doctoral research (2015):

> This lack of interest really worries me. Sometimes, I demand a lot of myself and ask myself: How am I supposed to give my classes? Isn't it important to awaken this interest? My colleagues often criticize me about this because I think one way and they think another way. Some tell me to let it go, that that's the way it is, it doesn't matter if they are getting bad grades, they even say that with time they [the students] will recover. But I cannot think like that, I want to know and investigate the reason, participate a little more in the life of my student, not only in school life. (Teacher, 2013 apud Franco, 2015, p. 115).

The teacher then identified the class of students who would participate in the research. According to her, they were the most apathetic to study and had numerous difficulties in the area of mathematics. During three semesters, the teacher participated in readings, studies, and dialogues on the theoretical and methodological contribution of the cultural-historical psychology of activity (Puentes; Longarezi, 2017) in order to appropriate it for instrumentalization and objectification in her pedagogical work. This process generated in the teacher contradictions and conflicts in her way of organizing pedagogical activity and resistance movements in search of overcoming them. To this end, Franco (2015) and the teacher dedicated themselves to the study of the modes and conditions of the class, in an attempt to seek the general mode of action of teaching, based on the cultural-historical psychology of activity (Puentes and Longarezi, 2017), that is, in the didactic elements that drive learning and development in a dialectical approach. Throughout the didactic-formative intervention, the teacher was able to constitute new needs, in close relationship with the motive oriented and linked to the objective content of teaching (Franco et al., 2019).

From the didactic-formative intervention, several tasks were organized to guide students in the appropriation of the essence of the concept of fractional, linear, quadratic equations and functions. The tasks in this system helped students to identify the internal movements present in the constitution of the concept of fractional equation, for example: **equivalence of values; range of variation; interdependence; and transitivity relations**. In the task for studying the concept of linear and quadratic equations, actions were provided for students to reflect on the general and particularized movements of the concept, such as: **fluency; interdependence and equality relations; properties of geometry and arithmetic.** In the tasks for the study of the concept of function, actions were foreseen for the students to analyze the interdependence relations between magnitudes, i.e., **range of variation; quantitative regularities.**

In the whole of the movement, 13 systems of study actions with the three concepts were developed. The correlations between motive, objective, and object of scientific knowledge in each system conducted by the students were captured through the analysis of the reflective written and verbal records of each student during the classes, all duly recorded and analyzed.

In turn, the teacher's training meetings with the researcher were recorded through audio recordings, in the researcher's field notebook, and in the teacher's reflective notes. From the reading and analysis of these records, objective and subjective products of her formative process were apprehended. The data from the totality of the teacher's formative process, under this didactic and dialectical approach, some analytical units were deduced: sharing/interactions; attribution of meaning; appropriation/objectivation, which, according to Vygotsky (1991), represent the universal-singular-particular movement as indivisible elements of the investigated essence. In the research, such units were evidenced by the episodes, because they reveal "[…] the nature and quality of the actions […]" (Moura, 2000, p. 60) of the teacher.

To meet the specificities of this article, we chose to present an episode with two scenes, in which there are some indications of the teacher's resistance movements and the search for overcoming them, as follows: "The difficulties of organizing the pedagogical activity beyond the formal logic" and "The analytical thinking in the construction of the concept". The data from these episodes were extracted from the training meetings (audio recorded) and from the teacher's reflective notes (written by her at the end of each lesson). Thus, some revealing indications of how the actions of the analytical thinking (analysis, reflection, synthesis, comparisons on how to form the theoretical thinking in the student, in the act of its achievement—praxis) constituted form and content of the didactic and formative intervention were apprehended.

12.4 The Development of Mathematical Conceptual Thinking in its Unity with Teaching Professional Development

The didactic and formative intervention was developed in interrelated moments: (i) creation of collective needs arising from the theoretical and practical activity of educational work; (ii) organization of elements of orientation and execution capable of providing the satisfaction of collective needs (teachers and students); (iii) planning of actions and study tasks for new appropriations and objectifications in educational work: organization of pedagogical activity under a developmental approach; and (iv) dialectical movements of overcoming between practice, experience, pseudo-concept and theory, the concrete thought and the generalized, in new situations of organization of pedagogical activity.

From the perspective of developmental didactics, "[…] the discovery of the essence of a given concept must be carried out by the student himself, through the actions deliberately and previously organized by the teacher" (Longarezi and Franco, 2016, p. 545). Here is a new need, constituted in the concrete reality of the teacher's teaching and in the context of research, through didactic and formative interventions. What actions help students to appropriate a theoretical/scientific concept and, at the same time, help them to develop their thinking in front of the concept, in view of its objectification in the reality of their social practice? His pedagogical work?

In the course of the didactic-formative intervention movement, the teacher began to be aware that it was not enough to list the school content to be worked on, and it was not enough to explain to the students the formal definition of the concept or the way to solve a certain mathematical situation, because it became necessary to guide the students to develop a way of thinking about how a certain theoretical concept is constituted (Franco, 2015; Franco et al., 2015, 2016, 2017, 2018).

In this way, the guiding elements of the psychological structure of the activity—**need, object, and motive—**gradually constituted the correlations with the elements of the execution: **actions, operations, and goals,** so that a new thinking and acting could be objectified in the social practice of the teacher, and consequently, in the learning activity of the students. The **actions** carried out by the teacher were: (1) reading texts based on the cultural-historical psychology of activity (Puentes and Longarezi, 2017), which includes concepts such as: education, learning, development, didactics, dialectical movement, formation of mental actions; (2) planning and didactic development of study actions, including algebraic concepts (in a process of confrontation between traditional didactics and developmental didactics); and (3) self-evaluation (Franco, 2015; Franco et al., 2019).

In turn, within the same psychological structure of activity, the **operations** related to the way such actions occurred: (1) collective and individual studies; (2) meeting to prepare and plan the process of mental assimilation of concepts (pedagogical actions that organize the study actions); (3) records of the meetings by the researcher; and (4) reflective notes by the teacher on what was done.

Therefore, the **objectives** of these actions were: (1) to analyze the theoretical-methodological assumptions that support the organization of the pedagogical activity under the cultural-historical psychology of the activity (Puentes and Longarezi, 2017); (2) to identify the theoretical-methodological assumptions of developmental didactics in the study actions; and (3) to organize the process of formation of mental actions to be performed by the students, according to these assumptions (Franco, 2015; Franco et al., 2019).

It is noted that the description of the functioning of the movement of this internal psychological structure of pedagogical activity aims to contribute to the formation of active and creative subjects of their own development. Always mediated by theoretical-methodological references and by social practice, aiming at the most humanizing and less alienating appropriations and objectifications for teachers and students. For Leontiev (1978), when the need is satisfied in the object, it is at a qualitatively ascending psychological level; therefore, the objectified need is understood as the reason, the one that forms the personal meaning of such activity.

In order for the reader to understand how this movement took place in the pedagogical activity, in its unity, with the study activity, the context of the episode "The difficulties of organizing the pedagogical activity beyond the formal logic" will be presented with the analysis of two scenes.

12.4.1 Episode a: The Difficulties of Organizing Pedagogical Activity Beyond Formal Logic

This episode occurs in one of the training meetings of the researcher and teacher, whose discussion focuses on the formation of theoretical thinking in the learning of algebra, under a cultural-historical approach. Several readings raise doubts, uncertainties, and apprehensions in the teacher in face of what is established in the school program. A conflicting situation is established that provokes new needs in her teaching action and makes her look for ways to solve it.

Scene a.1 (AOE-II-Date: 10/16/2012)

Subject	**Dialogues**
Researcher	After all, how do students form the concept on this basis, from abstract to concrete?
Prof	I surveyed the students about the way or manner in which they had studied, in previous years, the equation Did you use multiplicative principles, or addition and subtraction, multiplying, adding, or subtracting, or did you want the direct process? How did the teacher teach it last year? Was it all over the top like this? It was the direct one. Not me. I explain both, for them to choose, and leave them the option to choose. Before passing to the negative side and passing to the positive side, the exchange of signs, I could have used the principles, but I questioned them, and they said that they learned by the direct process "inverse properties". I said great! For me, it is much easier. So, what do I do? If I keep going back to last year's principles and subjects, I won't be able to continue with the content…
Researcher	I understand your concern, really, we have seen students continue their schooling, going from one year to the next, without understanding the concepts, because many times they can't use them in other situations different from the one worked on, or given in the classroom. That's right, the teacher teaches the formal language, uses the symbols of mathematics, but the students can't understand the reason for this use, there is a formalism of the content, the representation or the symbolic language is overvalued in detriment of the semantic language, the one in which the student understands the use of a certain formula by understanding the essential characteristic of the concept that is being formed. Therefore, they themselves prefer not to change either
Prof	You know, if the teacher doesn't comply with what is established in the CBC program, if he doesn't work with everything in the textbook, he is not considered a good teacher. The systemic evaluations also require compliance with the content. You know that within the school, among parents and even among co-workers, there is a culture of comparing what each one develops in the classroom, to what extent this or that content has been worked. The notebook where the student records what was done in the classroom is a reason for comparison for certain parents and supervisors. So, there is a pressure from the educational system, the school, parents and colleagues that it is difficult…
Researcher	Well, let's think about what is in these documents. Themes, content, skills, objectives are determined, but there is no didactic-methodological concern. Who is responsible for this decision? The teachers! But based on what principles does the teacher make this choice?
Prof	Generally, based on the material we have, the textbook and the CBC, and day-to-day experience in the classroom
Prof	And how does the textbook approach the content? What teaching methodology? As we can see, in this adopted textbook, it is taught in formal language, by mechanical training of the rules and resolution forms, without any active action by the student, without a methodology that helps the student to think about the concept. Do you think students would be able to act in a different way than this?
Prof	Yeah, in this class we have middle and upper middle-class students coming to school with the intention of playing, because they know that afterwards their parents will hire a private teacher to solve their difficulties… So, they don't get too involved, they don't ask questions about what they don't know, and this makes it difficult for the teacher. But I think we can try…

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Subject	Dialogues
Researcher	Let's try to go from semantic language to symbolic language, with understanding of what you are doing. This is the didactic principle. How to get to this point? It requires teaching methodology. We will try to work through the problem-solving method, where the student will construct, with the teacher's guidance, the characteristics of the concepts, giving them meaning. We can think of a work dynamic that involves the different groups and the collective to see how they are grasping the concept, and at the same time, we can perceive the movement of the reason before the study
Prof	All of this is very difficult for me. Although I always worry about teaching well, I confess that I hadn't thought about how the student learns. I am confused… In my undergraduate mathematics degree, I didn't study this, everything was taught in a traditional way. So, the same way I learned, I started to do it with my students, because even in the continuing education courses there is no such concern. If the teacher knows the content, then he knows how to teach ***Sources*** PUENTES & LONGAREZI (2013) **Theoretical Principles for a Developmental Didactics;** SCARLASSARI (2007) **A study of difficulties in learning algebra in differentiated teaching situations in 6th grade students**

Source Franco (2015, p. 186–7).

This episode refers to one of the training meetings on algebra learning under the bases of cultural-historical activity psychology (Puentes and Longarezi, 2017) and developmental learning, prompted by the research findings of Scarlassari (2007) and Rosa (2009). At that time, it was discussed how these two authors worked the problem situations with the students. The operation (reading a text) and the analysis actions performed by the teacher to "[…] raise the principle or universal mode […]" (Semenova, 1996, p. 166) of the concept of equation at the theoretical level are related to the object content of the activity (organization of pedagogical activity), by the analysis of other experiences of the mode of performance (Franco, 2015; Longarezi & Franco, 2016, 2017; Franco et al., 2019).

That textual discussion triggered in the teacher a conflicting and contradictory situation. For when analyzing the movement of development of concepts inherent to the field of algebra and also their social significance, as well as the level of generalization and abstraction that such concepts involved, she realized that teaching only the formulas of algebraic representations and their resolutions did not enable the student to understand the internal characteristics of the concept (Franco, 2015; Franco et al., 2019).

All this generated the teacher a lot of discomfort and feelings of contradiction with his way of teaching, because although he had always explained the use of the formula, he had never thought about how the student appropriated theoretical concepts. That is, she had never thought about how he could teach them to mentally operate on the internal movement present in this formula, in the theoretical concept under study, because, according to his point of view,

> The student will only assimilate if the teacher has the patience to teach [...] As for teaching, I think it is a program that I have to comply with, the content is the planning of the concepts

> themselves, and the development is that I can develop in the classroom the minimum so that my student learns (Teacher, 2012, apud Franco, 2015, p.166).

The excerpt shows that her conceptions of education, learning, and development clashed with the assumptions of cultural-historical psychology of activity (Puentes; Longarezi, 2017) and developmental didactics. This contradiction placed her in front of a conflicting problem that she needed to address, given her need to understand how this mode of acting would be and the need to comply with the curriculum program of the school context. "[…] So, what do I do? If I keep going back to last year's principles and subjects, I won't be able to proceed with the content" (Teacher, 2012, apud Franco, 2015 p. 188).

Her argument revealed how, in schools, the pedagogical concern often focuses more on learning the curricular contents than on the process of guiding the student to operate mentally with the concepts. In this case, the teacher was faced with a didactic and formative problem that only her knowledge and teaching experience were no longer enough to help her overcome this contradiction.

At that moment, it was understood that the **exchange and interaction** between the teacher, the researcher, and the concrete reality favored the formative movement. In this context, the dialogue and the analysis of the pedagogical practice favored the confrontation between the way of organizing the pedagogical activity, focused on empirical and theoretical thinking. They also made it possible to relate them to the dilemmas of the structure of the official curriculum of Minas Gerais, experienced by teachers and students in the school. The need to think about how to organize this process to guide the student in the theoretical appropriation of the concept began to guide their actions in the didactic and formative intervention, throughout the research. Therefore, the interaction between subjects who share together their problems can allow them to establish common goals to solve them (Franco et al., 2019).

In this movement, the emergence of a teaching need has been noted, guided by an internal interest related to the content and purpose of education: the didactic and intentional organization of pedagogical activity aimed at development. This element can promote the attribution of a personal meaning to the social meaning of the pedagogical activity, not as an end in itself, but as a necessary condition for the students to be involved (to have motives) and to develop thinking and new attitudes toward the object to be known. This aspect can be seen in the following excerpt from the teacher:

> [...] After this reading I organized a class (with another class of students not participating in the research) talking initially about this movement of algebra in life, relating it to the corporeal limit of man, to dance, to the movement of things in the world, in the universe (Teacher, 2012 apud Franco, 2015, p. 193).

The didactic and formative intervention enabled the shared activity and awakened the teacher's view to the internal aspects of the learning method, highlighted by Klingberg (1978, p. 292), as those procedures and logical operations are more likely to boost the creative and independent activity of students. Thus, in the conscious and voluntary mastery of the way of organizing the process of forming mental actions in the students, the teacher was gradually able to develop psychically and exercise her creative potential at work, in her pedagogical practice. Still according to the teacher.

> In the mathematics undergraduate course, I didn't study this, everything was taught in a traditional way. So, the same way I learned I started to do with my students, because even in continuing education courses there is no such concern. If the teacher knows the content, then he knows how to teach (Teacher 2012 apud Franco, 2015, p. 190).

In particular, it can be seen that if the conflict generated in this context was not mediated by the internal characteristics of the concept, those can form the teacher's theoretical thinking about developmental learning, and if it was not developed through actions, logical and abstract forms of this type of thinking, it could only result in the mechanical reproduction of this information. The words spoken by the teacher show this internal process of formation of her conceptual thinking, about her pedagogical activity, which is focused and oriented on a certain type of mathematical conceptual thinking of the students: "[…] It is interesting to see that the examples she uses in this research are intended to help the student understand the meaning of this symbolic language used in equations" (Teacher, 2012, apud Franco, 2015, p. 192). The teacher began to identify in herself her own movement to form conceptually a new approach to education, the actions taken and necessary to appropriate this theoretical perspective. Then, she experienced the process of revising the knowledge and mathematical concepts acquired during her educational and professional career. From the perspective of a teacher:

> I know that many things that I have taught until today were very mechanical. But we were educated that way. I didn't learn to think about the concept and much less teach the student to do this. We see that the Greeks didn't think like that. Why is it so difficult for the student to understand the symbolic language of the equation? (Teacher, 2012 apud Franco, 2015, p. 205).

It is observed that the teacher began to become aware of both her alienating conditions in front of the theoretical concept and her potential to overcome them: "[…] So they need to understand that an equation represents a situation of equality, equivalence, and that there is also another very important characteristic, the field of variation" (Teacher, 2012 apud Franco, 2015, p. 195).

The theoretical and methodological studies of cultural-historical psychology of activity, linked to the needs of learning mathematics, function mediating tools between the reality experienced in the teacher's classroom and the object of knowledge. In Franco's research (2015), it was argued that this double movement, practical-theoretical and theoretical-practical, with studies of cultural-historical psychology of activity and pedagogical work conducted inside and outside the classroom, helped the teacher to attribute new meanings to the social meanings of the educational processes. This movement made common activity possible and, it seems, awakened the teacher's view to the internal aspects of the **learning method,** as Klingberg (1978, p. 292) points out.

The formative meetings were constituted as an operation and were not exhausted in themselves, because they began to compose another relationship with the actions necessary for the organization of the pedagogical activity. Among them, analysis, reflection, synthesis of the general mode of action with theoretical concepts that began to be actively conducted by the teacher during the didactic and formative procedure. These actions, developed in a participatory way and through the collaborative elaboration of pedagogical activities, offered the conditions for the teacher to form new relations with the students.

The next ***scene A 1.2*** of the same episode A, "*The difficulties of organizing the pedagogical activity beyond the formal logic"*, showed how, through certain operations, new formative needs emerged that triggered new actions in order to achieve certain objectives. This systemic movement correlated with the object of the study activity: organizing the process of forming mental actions and developing meanings.

Scene A1.2 (AOE-II Date: 25/10/2012)

Subject	Dialogues
Prof	After this reading I organized a class (with another class of students not participating in the research) talking initially about this movement of algebra in life, relating it to the corporeal limit of man, to dance, to the movement of things in the world, in the universe. I gave the example of the classroom as a universe. So, if we want to find out in this universe the number of students in each of the four rows, we can think of a variation field of this situation (the possibilities, the variable), we can also call x, or unknown quantity, that which we don't know, or that which we want to find out about this universe, this totality. So, we went on building the equation, it was a very productive class, they loved it…
Prof	Will we have to use many lessons with this class to work on this concept? Will we have enough time in the fourth semester to finish all the content that is in the book? You know how these things are inside the school… The students compare what each teacher is working on in class or not, and the parents also demand this
Researcher	Students will construct the concept you are wanting to demonstrate, but in another way, where students can give meaning to a given situation involving the equation What do you think about thinking about developing a problem situation where they can represent reality and assign meaning to it, putting students inside the situation, making them think about meaning of equation?
Prof	Once I tried to demonstrate the equation by means of the tug-of-war game situation representing the equivalence and equality issue of the situations. But in the end, I don't think that was very helpful. They liked it, but soon forgot about it
Researcher	This exemplifies the need to develop specific actions of analytical thinking, which does not stop at the external appearance of the concept, only at perception, but goes beyond it, seeking to analyze its essence. Therefore, the logical actions will help the student to do this, by the analysis and reflection of what is part of this concept, what makes it be, what it is. So, what do you think about first designing a problem situation where he can think and try to solve it by himself, individually using another language to express the movement, without using the formal language of algebra?
Prof	Yeah, because of these textbook problems I don't think we are going to find what we want; they are not going to meet this objective. So, they need to understand that an equation represents a situation of equality, equivalence and that there is also another very important characteristic, the variation field

(continued)

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Subject	Dialogues
Researcher	Let's look at these examples of activities used in Scarlassari's research (2007) "write equations in symbolic language from rhetorical language", that is, represent the situation in another way without using formal mathematical representation. Another example: "pass into mathematical, rhetorical, formal language the movements below"; here the teacher's intention was to pass the idea of the movement present in the situations
Prof	It is interesting to see that the examples used by her in this research are intended to help the student understand the meaning of this symbolic language used in equations
Researcher	Let's think about a problem situation in the dynamics individual; group; group-class so that the students build the concept through the movement of the actions they are doing
Prof	Well, I think that based on what we are studying here, I don't see much sense in the lesson that I prepared before, you know, in that formal way without meaning, mechanically. I don't know now what I'm going to do until I start the intervention I'm thinking that I'm going to work with the students on the symbology that involves the concept of equation, I'm going to work on the meaning of the theory with them, starting from what they will need to understand the internal characteristics of the equation. I think that they were, and I too, acting very mechanically. So, I will get from themselves what they already know, even if they have seen it in a mechanical way
Researcher	They need to be able to identify the internal characteristics that define the concept: what it is that defines an equation to be an equation and not something else, and then understand the formal language
Prof	Students really enjoy looking for answers to challenges and comparing. For example: In that equation given in the book, who would have thought that from fractions, equations that are so different, we would really get to what the understanding of equation is: the equivalence between situations. It can be represented differently, but at the same time, it represents equality. I'm going to try to work out some problem situations in that approach
Researcher	What do you think about talking to the students, before we start the intervention, about the movement we are going to make with them, so that together we can define what we are going to need to achieve mastery of the concepts, establish objectives and goals, define our actions (teaching) and their actions (study/learning), so that everyone (students and parents) can identify the relationship of what we are going to establish in this movement with the school program, and the contents that are in the textbook. That is, to collectively build a roadmap to guide our actions, so that not only we know the path, but also, they and their parents can follow what we are going to do to achieve our goals (Texts: Scarlassari (2007) and Rosa (2009) Learning 2nd degree equation—An analysis of the use of developmental learning theory; LIMA, Luciano; Takazaki, Mário; Moisés, Roberto P. Equations: the movement particularizes, Belenzinho. [n.n])

Source Franco (2015, p. 192–3).

Such shared and individual syntheses opened paths to the understanding of conceptual nexuses, understood "[…] as a link between ways of thinking the concept"

(Sousa, 2004, p. 53). The aforementioned author states that fluency, variation field, and variable are the nexuses that form the concept of algebra.

In this path, Franco (2015, p. 269) clarifies:

> [...] in the case of the concept of 'function', the student appropriated the essence of its movement, starting from its general features (interdependent relations between two quantities that vary, and quantitative regularity), and from these he managed to extract its particular features (movement between the dependent and independent variables, the type of quantity to which they refer: segment or area). Considering Vygotsky's assumptions (2001, p. 269), we understand that for most students, these concepts developed in the field of algebra, mediated by previous generalizations of arithmetic and geometry. This happened because the logical operations with these concepts are seen as particular cases of a more general concept (Algebra), and the logical operation of thinking with them, in this case, will occur more freely.

In Franco's research (2015), it was observed that in the field of algebra, the teacher first had to understand and appropriate the essence and the connections of the concept and then guide her students in this theoretical appropriation. From the indications presented in the episodes, the teacher was able to make movements of theoretical appropriation about teaching, in its didactic and formative elements, although in the midst of contradictions, curricular limitations, and educational legislation in force in the country, as well as in the socio-cultural conditions of the students.

As a result of the didactic and formative intervention developed with the teacher and the students, some didactic actions were identified, which promoted the development of the subjects, which, in a way, help to compose the discussion of teacher and student training as a theoretical-practical and content-form unit. Within the framework of the conducted research, some didactic actions have been taken:

1. Organize teaching by providing students with the conditions, the tools of culture and the modes of logical and specific mental actions, with a view to the appropriation of theoretical concepts.
2. Analyze the concept, discover its essence, its internal logic to establish purposes and plan the actions that enable both guidance and regulation of the process, in a clear and active way by the subject in training itself.
3. To promote the autonomy and creation of the teacher in front of the analysis processes of his or her pedagogical practice, that is, in front of the preparation and organization of a teaching that promotes development.
4. Consider the system of relationships in the school context and the interests that arise from this field, which affect the cognitive, volitional and affective aspects of the subjects (Franco, 2015, p. 274).

The didactic actions cannot be seen as a prescription or technique: They constitute results learned from the movement conducted with certain subjects, which highlighted as a starting point the knowledge, experience and life of the teacher and students, historical subjects in a specific school and cultural context. By considering this starting point, the didactic-formative intervention triggered the teacher's confrontation movements with herself, with reality and with theory, to overcome the dichotomy between being and essence, in the propitious space of schooling and humanization, the school.

12.5 Final Considerations

In this text, we have tried to discuss the formation of mathematical conceptual thinking of a teacher and a student, as a theoretical-practical and content-form unit, as well as to understand under which conditions it would be possible to establish other relations in the internal structure of pedagogical and study activities in Brazilian public school contexts, having in mind the development of new meaning-forming motives, as oriented by the Leontievian conception. It is noteworthy that the didactic and formative intervention was mediated by scientific concepts about teaching in its relation to educational work. In other words, this process was constituted considering the unit of education-learning-development. The confrontations and conflicts about the process of scientific knowledge, coming from formal logic and dialectical logic, led the teacher to search for referents, aiming at the formation and development of new forms of conceptual thinking. This, in turn, was constituted in the sense of theoretical-practical and content-form unity in the education of teachers and students. These dimensions did not exclude each other in the process, but operated differently among themselves, in a movement that took place from the abstract to the concrete, in a dynamic and dialectical relationship.

The teacher's analytical actions on the research developed in the field of mathematics, specifically in the learning of algebra, based on developmental didactics, cultural-historical psychology of activity, historical-dialectical materialism and classroom reality, provided its materialization in the process. This materialization occurred through the activity itself (pedagogical work), through the interactions between subjects, theory, and reality. In such a way that this action, articulated with the verbalization actions about the objectives of the pedagogical activity and the research, favored the moments of synthesis and theoretical appropriations, having in view a more humanizing objectifications of herself and her students in the concrete reality of the classroom. When, in the didactic and formative intervention, the teacher took her own pedagogical practice (its material form), as well as the representation she had of it (its materialized form), she was able to think and reflect on it conceptually. This allowed the teacher to constitute instruments to put herself in a state of activity that put her in a process of rupture-development.

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Chapter 13
Contributions of the Cultural–Historical Theory to Activities for Teaching Measures of Time and Area

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Abstract The development of the measurement concept in school depends on the determination of the quality to be quantified. In this appropriation process, there are three essential phases: to highlight the magnitude to be measured, to compare magnitudes of the same nature, and to express numerically the result of the comparison. This article approaches the work with magnitudes that, perhaps, are considered the most complex to be understood, especially by children: the magnitudes of time and area. It has as its main goal to point out the elaboration and discussion of teaching situations for the concepts of measure of time and area, revealing and quantifying the quality they express. Based on the cultural–historical theory and from the assumptions of the Teaching-Orienteering Activity, the purpose is to discuss a non-procedural teaching practice that transcends the reading of hours and calendars, the application of meaningless formulas, and dialogues about possibilities of development of work with measurement in the early years of Elementary School, through the development of learning triggering situations.

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13.1 Introduction

The treatment given to the learning of measure of time and area was one of the study objects of the project "Math Education in the early years of Elementary School: principles and teaching organization practices" (2011–2014),[1] coordinated by Prof. Manoel Oriosvaldo de Moura, Ph.D. in Universidade de São Paulo/USP, linked to Programa Observatório da Educação (OBEDUC), and financed by Coordination for the Improvement of Higher Education Personnel (CAPES). As one of the results of this activity, a fascicle with activities for the teaching of magnitudes and measures (de Moura et al., 2018) was elaborated, in which one of the chapters is dedicated to the development of time measurement and the other to area measurement.[2]

The movement of the actions to be highlighted here emerges from the work developed during the mentioned project and that, according to the members, represented an important research experience that considered school as a place of formation and learning, both for teachers of basic education and researchers. Such project was materialized as a result of partnerships established within the group GEPAPe—Grupo de Estudos e Pesquisas sobre a Atividade Pedagógica.

Relying on the theoretical–methodological foundation of the Teaching-Orienteering Activity (TOA), we highlight our considerations regarding the elaboration of teaching activities, considering aspects of human creation that develop throughout history and the way they shape our culture. In this context, contrary to what is expressed by an idealistic view that considers that mathematics is born from refined thought, we understand mathematical concepts from a historical perspective, as a result of practical needs of the social life, and, therefore, as a social object, a knowledge in constant development (Alexandrov, 2016).

Thus, we believe in the possibility of enriching the work with the concept of measurement of time and area in this proposition to the early years of Elementary School, based on a "non-procedural" teaching concept and that does not have as its unique goal the development of techniques of reading hours, filling calendars, and area calculation only by using formulas. This conception emerges from the fact that those techniques and formulas are only concept syntheses, logically organized, but does not explicit the long human process of construction and the relation they have with others concepts, evidenced by its nexus.

The central objective of this text is to discuss possibilities for organizing the teaching of concepts of time and area measurement in the early years of schooling

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[1] The project constitutes itself, as a great collective, from the organization of four research centers, located in the following institutions: Faculdade de Educação da Universidade de São Paulo (FEUSP), Universidade Federal de Santa Maria (UFSM), Faculdade de Filosofia Ciências e Letras de Ribeirão Preto (FFCLRP/USP), and Universidade Federal de Goiás (UFG). The teaching situations presented here were elaborated and developed collectively by members of São Paulo's core, at FEUSP.

[2] With the work developed during the project, four e-book volumes were produced, with the general title of "Mathematic Activities in the early years of Basic Education". Each volume is composed by a courseware covering the following themes: statistics, measurement (time, area, length, mass, and volume/capacity), numbers and operations, and geometry. This material can be found at http://www.labeduc.fe.usp.br → LABMAT → OBEDUC.

through a proposal based on the perspective of the cultural–historical theory and guided by the theoretical–methodological principles of the Teaching-Orienteering Activity. In that regard, pedagogical intentionality is considered a fundamental element for teaching organization and becomes essential to the educational practice, as much as the teacher's explicitness regarding the essential elements belonging to the concepts to be taught. Thus, the unit relationship between these two cited elements, intentionality and mathematical knowledge, will be highlighted in the presented activities for teaching time and area.

Based on the theoretical–methodological foundation of the Teaching-Orienteering Activity (TOA), we highlight our considerations in relation to the elaboration of activities for teaching, considering the aspects of human creation that develop throughout history and the way they shape our culture. In this context, contrary to what expresses an idealist view that considers that mathematics is born of pure thought, we understand mathematical concepts, in the light of the cultural–historical theory, through a historical view, the result of practical needs of social life and, therefore, as a social object, impregnated with the human condition. An ever-developing knowledge (Alexandrov, 2016; Caraça, 2010).

We have the understanding that, from the beginning of schooling, when the child's first relationships with mathematical theoretical knowledge occur, this must be meaningful to him. Its knowledge begins to expand, without forgetting what he brings into his life story. The teacher, through the teaching activity, has as main objective to enable the appropriation of concepts considered relevant for the formation of his students, developing activities for the teaching of mathematical concepts based on the perspective of a teaching that stimulates the psychic development and the abilities human beings, as evidenced by the studies of Vygotski (1991) and Davídov (1988).

For this reason, we defend learning to be meaningful, and for that, we believe to be necessary for the child to appropriate mathematical knowledge by doing mathematics, in activity, where teaching propositions that are related to social and cultural practices are structured, human, and historical (Moretti & Souza, 2015).

13.2 Organization and Development of the Process in Activity

When we understand the need for a way of teaching that has, intentionally, an organization that allows the promotion of human development, we follow what de Moura (1996, 2004, 2017) proposes as a Teaching-Orienteering Activity (TOA), understood as a theoretical–methodological basis directed to the teaching and learning processes and which is based on the perspective of the cultural–historical theory. The theoretical–methodological principles in that proposition

> [...] explicit as a unit between the teaching activity (by the teacher) and the learning activity (by the student) in the pedagogical activity context. This activity, according to the author, must be organized in a way that allows the interaction among the subjects that, facing a problem situation, share the signification of necessary concepts for the problem resolution

> that mobilize them to solve it (de Moura, 2012). Thus, it enables that the subjects attribute signification to their actions during the appropriation of the social significances developed by the humanity experience and synthesized in the concepts (Munhoz & de Moura. 2020, p. 357).

As exposed by de Moura et al. (2010a, 2010b), the Teaching-Orienteering Activity keeps the structure of the activity proposed by Leontiev (1978), because it presents a need, a motive, objectives, and actions that consider the objective conditions of the school. The option by the organization of teaching, founded on TOA's theoretical–methodological principles—as highlighted by Gladcheff (2015)—provides attention to the individual differences, the particularities of the problem set in action, and the diverse knowledge present in the educational context, which emphasizes the responsibility of the teachers, since their intentionality is present throughout the process. It is considered orienteering because

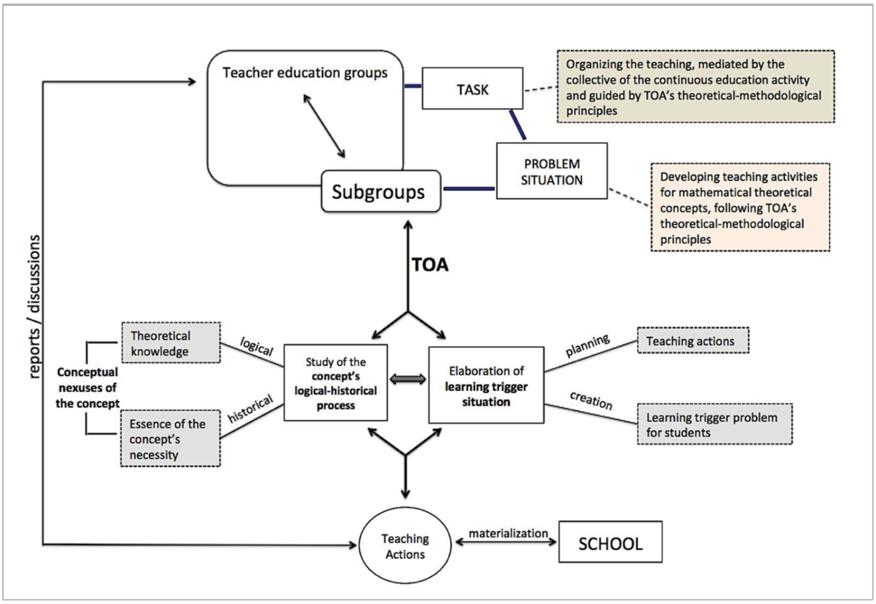
> [...] it defines the essential elements of the educational action and respects the dynamics of the interactions that do not always reach the results expected by the teacher. It establishes the objectives, defines the actions, and elects the helping teaching instruments; however, it does not hold the entire process, simply because it accepts that individuals in the interaction share meanings that change in face of the object of knowledge under discussion (de Moura, 2012, p. 155, our translation).

The teaching situations to be presented here were elaborated and developed in the schools by teachers of Basic Education, who participated in the teacher education activity organized during the development of the mentioned process. They were planned in subgroups formed inside the formation environment and guided by the principles of the Teaching-Orienteering Activity, as the following picture illustrates.

By means of the movement systematized in Fig. 13.1, we represent how the members of the formation movement were organized for the solution of what became a situation problem for them: to develop teaching activities to theoretical mathematical concepts, following the principles of the Teaching-Orienteering Activity. Such movement allows us to systematize how we work with the formation of the teaching's theoretical thinking, possible to be revealed in the articulation between theory and practice, in the constitution of pedagogical praxis that involved them in a group activity.

In the systematization previously presented, the principles that govern the Teaching-Orienteering Activity and that served as mediating elements in the process are present. Therefore, the group was divided into subunits from four to six members, each one responsible for creating teaching activities for a given mathematical concept. The concepts we discuss in this article, as previously mentioned, are the concepts of measurement of time and area, worked by two subunits of the investigation center of the Faculdade de Educação da Universidade de São Paulo (FEUSP).

To start working with the mentioned concepts, the teachers started their studies, going through, themselves, collectively, the genesis that may be revealed in the history of the concept to be worked with their students in order to identify what we understand by conceptual nexus (or inner nexus) of the concept. Those, for their part, associate with the logical–historical movement of the studied object, representing the essential



Orienteering Activity (MUNHOZ; MOURA, 2020, p. 366)

Fig. 13.1 Movement of planning and development of teaching activities based on the principles of the Teaching-Orienteering Activity (Munhoz & de Moura, 2020, p. 366)

aspect of the concept and, in this case, contain "the logic, the history, the abstractions, and the formalizations of the human thinking in the process of constituting itself as human through knowledge" (Souza et al., 2014, p. 96, our translation).

Based on the understanding of this movement, understood as the logical–historical process of the concept, the teachers start to understand the history of mathematics as a supporting tool for teaching and establish a new relationship with knowledge, looking at science "as a living organism, impregnated with human condition, with its strengths and weaknesses" (Caraça, 2010, p. vii, our translation) and historically built as a product of social interests and needs. And it is with this conception, as an instrument in a symbolic tool dimension, that we understand it to be necessary for its appropriation by those who integrate the school. We believe in this possibility, considering the interface between the history of mathematics and education, beyond a historiographic view, searching, in the logical–historical process, the movement of thinking in the context of the formation of the studied concept.

Thus, founded on the carried out studies, a learning triggering problem is formulated as part of a learning triggering situation (LTS). The latter has a set of actions that aims to motivate the students so the concept is put in motion in order to be appropriated by them. Nonetheless, it is important that the mentioned learning triggering problem keeps the structure of a learning problem as defined by Rubtsov (1996), by which the student, in solving it, "gets appropriated of a general mode of action, which becomes a guiding base of the actions in different situations that surround them, and not a practical concrete problem that, for its part, seeks modes of action in itself and whose resolution only works in a specific, particular, situation" (Munhoz & de Moura, 2020, p. 369).

As claimed by de Moura et al. (2018), the learning triggering situation, through the given problem, will allow the students, when solving it, collectively, to appropriate a form of general action that may become the guiding basis for their actions in different situations surrounding them, and it is going to provide the appropriation of the genesis of the discussed concept.

The genesis of the concept, in turn, is associated with the conceptual nexus understood as the essential aspects of the concept and its determinants and that, in our case, make up the concept of measure and bring the logic of the production process of this given knowledge, according to Caraça (2010), through: identification of the magnitude of an object or phenomena to be measured (what is going to be measured); comparison between the two (or more) objects or phenomenon that have the same magnitude (establishing major and minor or equal); establishment of a common unit of measure (or pattern) that allows quantifying the magnitude more accurately.

By emphasizing the resolution of the proposed problem in a collective way, we base ourselves on the perspective of what Davidov (1988) and Vigotski (2009) demonstrate when they affirm that the appropriation of theoretical knowledge in the direction of the subject's development is considered the essential objective in the educational process, since, as we stated before, in order to appropriate a new concept, the subjects first relate to it through social activities (interpsychological) and later turn it to themselves (intrapsychological). That is, learning is in the collective.

The learning triggering situation, according to de Moura et al. (2018), may materialize itself in a game (with pedagogical purpose), a problematization of a situation of daily life emergencies (that makes it possible for the students to face the need to experience the solution of significant problems for them and that are in evidence) or a virtual story of the concept (which puts children in front of a problem situation similar to the ones faced by humankind, generally speaking). It is relevant to point out that the virtual story of the concept is not about a factual history, but a made-up history, with a ludic purpose, aiming to put children in the process of resolution of a problem, that, somehow, imitates the essential characteristics of the human activity that produced the concept. This is highlighted by Leontiev (1978) by approaching the concept of appropriation, claiming that, for a concept to be appropriated by the student, an activity that imitates the essential characteristics of human activity which produced the concept is necessary.

Therefore, the learning triggering situation must

> [...] contemplate the genesis of the concept, that is, its essence; it must express the need that led humanity to the construction of the aforementioned concept, how problems and human needs in a given activity emerged, and how humans were working out the solutions or syntheses in their logical-historical movement. (de Moura et al., 2010a, 2010b, pp. 103–104, our translation).

As a fundamental part of the learning triggering situation, teaching actions are planned to guide the students to the solution it offers, setting the concept in motion to be appropriated by them.

Figure 13.1 also shows us that, as an individual action of each teacher, what was collectively planned by the group is taken into their educational practice in order to "modify their students' thinking and also to learn and transform their knowledges and themselves, in a dialectic process" (Munhoz & de Moura, 2020, p. 369). Continuing the process, what had been experienced at school was taken to the teacher education group in order to, collectively, reflect on and evaluate the actions made, allowing, therefore, to make changes in the developed teaching activities.

13.3 Working with Concepts

As a result of the process, we present a synthesis of the triggering learning activities developed for the concepts of area and time measurement.

13.3.1 Area: Measure of the Amount of Surface

The need to measure is largely present in human activity throughout history. To control the variations of magnitudes as length, area, volume, capacity, mass, angle, and time attributed new qualities to the various activities, such as: agriculture, livestock, calendar elaboration, construction of containers, and location techniques (Hogben, 1956; Eves, 2004; Fraga, 2016).

Among the mentioned magnitudes, we are going to approach the area measurement, which has recordings of its determination since ancient Egypt. In those days, it was necessary to calculate the portions of land administered by the Egyptians, for part of the land was flooded by the Nile River, during the flood period. Thus, with the calculation of the new area, the taxes to be paid were updated (Caraça, 1951).

In the present days, we can verify the use of area in activities such as: projecting a building, covering the rooms of a residence, informing the size of agricultural property, planning the use of a given surface such as lands, fabrics, and paper, among other uses. Due to this relevance, area became one of the Basic School contents and begins to be approached in the early years of Elementary School—as predicted by the national curriculum documents (Brasil, 1997, 2018).

However, we observe in our teaching activity and teacher trainers that the teaching of this area is, on many occasions, related to the length magnitude and the formula (composed of segments of the geometric shape). That leads students to misconceptions, such as considering the area a linear quantity or mistaking it for the perimeter concept.

So, the organization of teaching does not prioritize the appropriation of area as a magnitude that quantifies the surface, which should be central in the process of appropriation of the concept, mainly in its introduction, in the early years of Elementary School.

Admitting area as a magnitude, we understand it as:

> [..] a quality of an object, or phenomena, that may be quantified. The quality of an object or phenomenon can be understood as a group of relationships we establish between these objects or phenomena. Therefore, the perception of quality is always relative to something, a product of comparison and identification (de Moura et al., 2018, p. 5, our translation).

Thereby, the appropriation of the concept of area as a magnitude becomes necessary, that is, as a quantifier of the surface. The national curriculum documents, when dealing with magnitude and measurements, highlight that "in Elementary School—early years, the expectation is that students recognize that measuring is comparing a magnitude to a unit and to express the result of the comparison through a number" (Brasil, 2018, p. 271, our translation).

Thus, for understanding the importance of measuring in basic schools, more specifically, measuring area, we propose to highlight it as a surface quantity, based on the cultural–historical perspective and based on the principles of Teaching-Orienteering Activity.

13.3.2 A Learning Triggering Situation Materialized in the Virtual Story of the "Vegetable Garden"

In a synthesized way, we present the following report, which talks about the learning triggering situation, elaborated by the collaborative group, made up of teachers of Basic Education, school coordinators, graduate students, and the project coordinator, to develop the concept of area, through a virtual story.

The elaboration process started with the knowledge present in the national curriculum documents about the teaching of area (Brasil, 1997, 2018) and research regarding the elements about the historical–logical movement of building this concept (Caraça, 1951; Eves, 2004).

After comprehending the genesis of the concept of area and how this magnitude presents itself in the early years of basic school, a virtual story that proposes the building of a vegetable garden was elaborated. In this virtual story, the learning triggering problem presents itself in the equal division of an irregular land among the groups of students.

The choice of a virtual story allowed the experience of constructing an imaginary vegetable garden for schools that did not have the resources to plan a real garden; however, in some schools, the existence of a vegetable garden is possible, so this task may also be developed as the building of a real garden.

The learning triggering problem to be solved by the students was the equal division of an irregular land, as shown in Fig. 13.2—this shape was determined intentionally. The researched teaching materials, brought by the teachers to be studied in the groups, indicated that the teaching of area often began by counting squares, associating them to the unit of area and the row-by-column multiplication. Another reported situation,

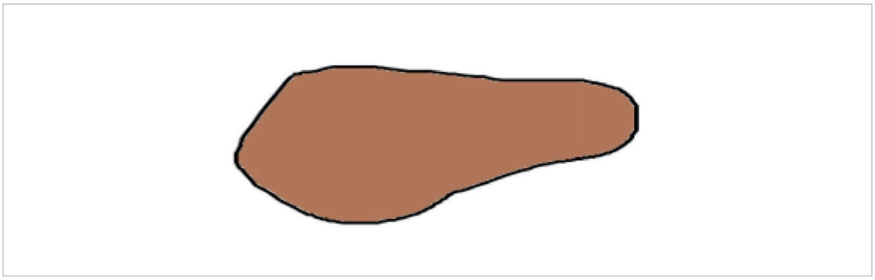


Fig. 13.2 Virtual story of the vegetable garden (de Moura et al., 2018, p. 104)

from the experience of the educators, was the recurrent association between multiplication and a squared design and between perimeter calculation and a highlighted outline.

> The comprehension of the calculation of area by multiplication and the association of the squares to the unit of area are recurrent difficulties for the students. Thus, it is necessary to develop the theoretical concept, related to the appropriation of the essence of the concept, for studying only the questions in squared patterns does not produce a movement that promotes a clear comprehension by the students but an empirical knowledge in which the students solve problems based on the visualization and observation of the drawing (de Moura et al., 2018, p. 103, our translation).

By proposing an irregular land, the intention was to make evident that, in measuring a surface, another surface is necessary as a unit of measurement and so unlink it from a possible relationship with its perimeter. The essence is in the comparison of magnitudes of the same kind and, later on, counting how much a given surface (adopted as a pattern) fits in a surface (to be measured).

In addition to highlighting the two-dimensional character of the area, the irregular shape provides a discussion of some reasons that influenced humanity to delimitate surfaces in polygonal shapes or their composition: ease and accuracy when measuring.

Subsequently, we seek to emphasize the reason for the squared (square as a unit of surface measurement) and the real meaning of multiplication (count of the unit of measure used).

The triggering problem presented (de Moura et al., 2018, p. 104) was the following:

We will build a vegetable garden on this land. For that, we need to divide it equally among the groups, for each group will be responsible for taking care of a part. How can we divide this surface so that each group has the same amount of place to plant?

The learning triggering situation was proposed for students in the early years with the primary objective of the students choosing the way to divide the land. By proposing the virtual story, the virtual land was presented—a cut fabric, shown in Fig. 13.3, where the students carried out actions letting them freely, organized in groups, choose the strategy they considered more appropriate.

Fig. 13.3 Division performed by students. The material used was cut nonwoven fabric and tape to mark the division

When developed in a municipal school, for example, a student suggested, and the others agreed, that each group took two portions of the land. They chose to divide the distance of two palms, that is, performed a linear measurement. This action may also be seen in Fig. 13.3.

The discussion, in groups, about how to solve the presented situation is one of the essential elements to build the teaching situation from the theoretical–methodological principles of the Teaching-Orienteering Activity. Vygotsky demonstrates that:

> [...] the most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge. (Vygotsky, 1978, p. 24).

The way in which students express the development of the solution to the posed problem, through social interaction, allows the educator to evaluate the construction of the concept, according to the cultural–historical theory.

After students finish the division, the teacher asks them if the division was fair, in case the question does not come up from them, during the work. In this case, the teachers can draw attention to the question and the group begins to evaluate the action carried out by them. During the development of the division, students noticed that some lands were bigger than others. They pointed out the use of the palm as a reference as the problem and proposed to solve the issue using a ruler. By testing, they noticed that the ruler did not help because, even though they equalized the distances between the tapes that they used for marking the division, it did not solve the problem of one land longer than the other.

One of the possible suggestions to be given is to cut the land, since it is made of fabric, which is represented in Fig. 13.4. This discussion is very interesting; after all, the task of measuring an irregular land is complex, and it was chosen for that purpose.

Fig. 13.4 Land cutting suggestion given by the students (Piovezan et al., 2014, p. 759, adapted)

This is a possible historical problem that created the need to regularize formats by drawing polygons and circles.

The teachers may interfere by saying the whole land can be put to good use for planting, and, to compare the division performed, they need to pay attention to the amount of land that fell to each group—to highlight the land surface, the quality to be measured, and the teachers may run their hand over the fabric, showing the students the totality of each group, removing the initial limitation of the distance between the tapes.

We noticed, in dialogue with the students, indications of the appropriation of the represented quality by the amount of surface. The search to compare with another surface is started, and they look for objects to perform the measurement (quantify the quality), paving the divisions of the land with books or dominoes.

Lastly, the need for everyone to use the same object arises (search for a standard unit) to make the comparisons of the performed measurements. This is the case represented in Fig. 13.5.

Therefore, during the development of actions planned for the learning triggering situation, we went through the objectives of the virtual story of the garden for students in the early years, namely: to generate the identification of the quality to be measured, in this case, the amount of surface, to seek methods to quantify, and to generate a need for the standard unit.

In the end, some considerations are important to be discussed in the collective: due to the shape of the land, its equal division is not possible, just an approximation, since a precise determination of the area is not possible; to systematize the carried out process highlighting the area as a surface measurement; and the human need of delineation of surfaces in polygonal or circular formats, due to the greater facility of calculating their areas.

Fig. 13.5 Group compares by using dominoes available in the classroom, the group uses books (Piovezan et al., 2014, p. 759)

13.3.3 The Measure of Time

School, as already defended by the cultural–historical theory, is the privileged place for the appropriation of the historically produced knowledge (Duarte, 2001; Saviani, 1991), being up to the teacher to intentionally organize the teaching, encouraging students to advance beyond their everyday knowledge.

In that regard, when we work with the concept of measure, we need to determine the characteristics of the object, or phenomena, we are considering, highlighting, thus, a magnitude to be measured.

In cases in which the quality of the object, or phenomenon, may be expressed in numbers, it means it can be measured. This argument is supported by Caraça (2010) when indicating that, throughout measurement, humans can numerically express the quality of an object or phenomenon. We must consider, however, that measurement is always relative; that is, it only exists in the relationship between objects or phenomena.

In our context, we bring to discussion the work with the measurement of one of the magnitudes that, perhaps, is considered one of the most complex to be understood, especially by children: time magnitude.

Working with the control of the variation of passage of time, that is, with the measure of time in a meaningful way, is not an easy task for teachers, for this is an intangible magnitude associated with processes and, therefore, is not materialized in physical objects. To be able to measure the passage of time, it is necessary that we do it in an indirect way, that is, by using other magnitudes.

Ponte and Serrazina (2000) explain this statement when reporting, for example, that time can be measured through the angle traveled by the hands of a clock, by the length that a candle burned, or by the amount of sand that fell into an hourglass.

As we can see, in these examples, we simultaneously use angle, length, and mass magnitudes. The non-comprehension of the complexity of this concept may explain the reasons for its teaching to be essentially based on procedural processes of time reading and calendar filling.

Through human history, we highlight, the understanding of Crosby (1999), that:

> For many people, time seemed an unfractionated flow. Therefore, the experimenters and inventors spent centuries trying to measure it, imitating its continuous passage, that is, the flow of water, sand, mercury, ground porcelain, and so on – or the slow and regular combustion of a candle away from the wind. But no one had ever devised a practical way of measuring long periods through these means. The substance in motion become gelatinous, freeze, evaporate or coagulate, either that or the candle would burn at a perversely fast or slow pace, or it would melt – something would go wrong.
>
> The solution of the problem became possible when stopped thinking of time as a regular continuum and started thinking of it as a succession of quantities (Crosby, 1999, p. 85—our translation).

This allows us to indicate that the ability to measure time does not develop until the child realizes that the events are separated by time intervals highlighted by the above quote, that is, as a succession of quantities. In fact, "the notion of 'time intervals', that is, the space of time occupied by an activity or the time that goes from one instant to another is a key idea that them has to realize" (Gladcheff, 2015, p. 196—our translation).

This is an understanding that the teacher, when starting the work with a measure of this magnitude, needs to have to enable a meaningful learning.

13.3.4 A Learning Triggering Situation Materialized in the Virtual Story "A Problem for the Curumins"

By recognizing the relationship in which the concept is taken as a social object, developed through a human activity triggered by a necessity, it is possible to understand that this concept may become of interest to children and, therefore, more significant. In that regard, the students must actively participate, which will allow their transformation, and it "does not remain only in the act of teaching/learning but for the individual's entire life" (Rigon et al., 2010, p. 32, our translation). Nonetheless, this is only possible when there is intentionality and organization of the activities of the teacher. We understand that the educational practice must be guided by the intentionality of forming individuals in the social direction of human formation that has the collective as a reference (Gladcheff, 2015).

The report we will do next shows, in a synthesized form, the learning triggering situation elaborated by one of the subgroups from the collective of teachers that participated in the teacher training activity carried out during the project mentioned in the introduction of this text. It is a virtual story of the concept, and it was elaborated to work time measurement separated in gaps, limited by unnatural events, and that makes use of instruments so that it can be observed (Gladcheff, 2016).

During the development of the story, children, mediated by the teacher's knowledge and actions, can comprehend the concept of subjective time (how its passage is felt), realize the need to control time objectively (given by an instrument of measure), and use an instrument, significantly, to perform a measurement even if not very precisely.

The story begins with the characters Raira, Apoema, Irani, and Raoni, who are curumins *from a tribe located in Parana and are playing near the river. Raira finds a red feather headdress. As they did not know which animal the feathers belonged to, they asked Apoema's father, who informed them that they belonged to the red guara, a rare animal in the region and a close relative of one of the sacred birds of Ancient Egypt, the ibis of the Nile river.*

The curumins *wanted to play with the headdress all the time. And that was when the problems started. They wanted to play together but there was only one headdress. In the game, whoever had the headdress would be the chief of the tribe, so everyone wanted to be the chief to wear the headdress. Hence the idea:*

Let us do the following - said Raoni - Each one of us keeps the headdress for a while and then hands it over to the other.

Everyone agreed and then decided the sequence of who was going to wear the headdress: Raira, Apoema, Irani, and Raoni.

While Raira was wearing the headdress, the other children were very anxious, as they could barely wait for their turn. But Raira did not want to let the headdress go and always felt like she had spent little time with it.

Wow! It feels like Raira has been with the headdress for so long - said Apoema.

True, - said Irani - Soon enough it will be dark and it looks like we will not be the chief.

Raira, then, decided to give the headdress to Apoema, who was really happy. But as time passed, in the same way, Apoema had the feeling of having played little with the headdress. But, to Irani and Raoni, it seemed the opposite.

We need to discuss and find a solution because Irani and I also want to be the chief today - said Raoni, a little crestfallen.

It was indeed getting dark. At the end of the day, Raoni and Irani ended up not wearing the headdress, not even for a bit. It was already dinner time and soon they would all go to bed.

Raoni and Irani were really upset, for they would have to wait until the next day to be the chief. Apoema and Irani, for their part, were happy, for they had used the beautiful and rare object. So, what do you think happened? Why did Raira and Apoema feel time passing so quickly and to Raoni and Irani time seemed to pass so slowly? And how can we help the curumins *so all of them can play with the headdress on the same day?*

In this intervention, it was possible to discuss situations experienced by the children themselves in the context of subjective time. In the second question, the idea was to make children realize, through the sharing of meanings of the concepts they already knew, that time could be divided equally for everyone. And, in this case, it was really important the fact that, in the fictional tribe, where the story took place, clocks were not known by the native people.

The next morning, after much discussion, the children came to an agreement and decided that they would share time equally.

All right, then, - said Irani - Let's share the right time each one will keep the headdress. But how can we do that?

And so, another problem emerges to the curumins*: how to share time and control it so everyone can play with it for the same amount of time?*

What do you think we can do to help the curumins

At this moment, the children could formulate hypotheses about how they could control time, without using the modern clock. After the hypotheses are formulated and tested, the story continues to its conclusion.

Wow! It's difficult, isn't it? - said Raira, lost in thoughts. Oh, listen up! How about talking to our shaman and seeing if he can help us? I bet he has a solution for that. Nice! - everyone shouted - Let's go!

Once with the shaman, the children told him what had happened the day before. The shaman said he had a solution for that. He went to the taba *and came back with a small object in his hands that had been given as a gift by a friend from the city and put it in the children's hands.*

That is an hourglass and it is used to measure time. It will help you with that problem.

So the curumins *took the hourglass and started to handle it to find out how it could help them to measure the time each one would spend with the headdress.*

What about you? Do you know how this instrument is used to measure the passage of time?

In order to explore and understand how the hourglass works, some types of this instrument were taken to the classroom so that the students could explore it in the time measurement context. Then, there was the socialization and synthesis of ideas.

According to the reports of the teachers who developed, at school, the actions planned for the learning trigger situation, while holding the hourglass, the children created a game called "hot potato with the hourglass." Sitting in a circle, the hourglass was passed by all of the children's hands, and it should reach the end of the circle before all the sand had gone down.

It was possible to perceive the children's enchantment with the instrument that was being used at that moment: the hourglass. Although they still did not have an understanding of how they could record and identify the "amount of time" that had passed from one moment to another, the object mobilized them in such a way as to create a ludic practice that allows exploring the concept that, at that moment, was in motion: time measure. The hourglass, as reported by one of the teachers, allowed children to observe the visible passage of time when materializing in the movement of the sand falling in the hourglass. That was how the children started to realize that it was possible to measure the passage of time,

After the children recorded, in the form of drawings, what they had experienced so far, a game called *Bolinhas ao cesto* (Balls in the basket) was used allowing them to make conscious use of the measurement instrument proposed. The game took place in a competition between teams of five students and balls (such as those used in ball pits), a big basket, chalk (to delimit the space for launching the balls), and an hourglass to measure the time of the game were used. The students of a team positioned at the same distance from the basket, delimited with the chalk, started to throw the balls, without crossing the line, as soon as the hourglass was turned. The team stopped throwing when all the sand had gone down and the balls in the basket were counted. The winning team was the one who threw, inside the basket, the largest number of balls during the time measured with the use of the hourglass.

The experience with counting time through an instrument to regulate the rules of the game showed that, in another situation out of the curumins' problem, such instrument, in this case, the hourglass, brings with it the generality of being able to measure time in various circumstances and enables the expansion of discussions aimed at the solution of the time of use of the headdress by curumins.

From this point, the interests also expand to another question, the precision of the instrument used and the need for its use, in addition to the improvement presented as an evolution of the measuring instruments present today. It becomes possible to explore the logic present in the working of analog and digital clocks, of the structuring and use of physical and digital calendars, and of other measurement instruments that bring within themselves the ways of thinking and producing knowledge, that is, the understanding of the process of meaning embodied in the objects resulting from human doing.

In this way, the situations of experiences provided by the learning triggering situation expand the possibilities of appropriation of the concepts that are intentionally explored in the action of a collective immersed in a context structured and designed for that, especially mediated by instruments when considered in their essence. Such reach demands access to higher levels of thinking with scientific knowledge as an interface.

13.4 Discussion

An important element for the teacher to feel safe when developing teaching activities is that he or she has clarity of the essential elements of the concept. By the presented learning triggering situations, situations are proposed in which the students set in motion the concept of measure, whether measure of time or amount of surface, in which they establish comparisons (larger, smaller, or equal) and define units of measure (number of squares placed in the space for the vegetable garden; amount of sand in the hourglass).

When they perform time measurement, they use the hourglass properly, which, in our understanding, allows the *materialization* of the origin of the concept of time measurement and allows the child to visualize the process of passage of time from the movement of the sand falling (Gladcheff, 2016).

In this case, students come across the need to develop actions that take to the theoretical concept of time measurement together, collectively. This means that there is a process of signification of the concept in the child because it takes place in the collective. The teachers, through the questions integrated into the virtual story, put the children in the need of the appropriation of the concept of time measurement for the problem to be solved.

In the virtual story presented, children are faced with problem situations, and to solve them, they need the mediation constituted by the teachers' actions and knowledge and the material resource, in addition to interacting with their peers. Through this, complemented by the game, children could understand that time can be divided and that there are instruments that permit this control.

Regarding the learning triggering situation for the measurement of amount of surface (area)—the virtual story of the vegetable garden—the students develop actions over a cut fabric in an irregular form, which is indicated as the land where the planting will be carried out.

It is in the experience, manipulation, and division of the land that the concept of area is put in motion to be appropriated by the students. The land is the instrument, mediating element of the relationship of the students with the concept, differentiating the area from the linear measurement, and, mainly, recognizing it as the quantification of a surface from another surface.

The proposed problem has an interdisciplinary character because, after the students complete the actions related to fair division, they may, for instance, look up some vegetables, put the research images on the part of the land that belongs to the group, and make an exhibition to the school. On the other hand, school units that have land may, later on, plant herbs for spices to be used in the school kitchen, among other possible alternatives.

With these actions, we notice the possibility of the child's psychic development, since he or she "starts to participate in a collective activity that brings new needs and demands from their new modes of action" (Sforni, 2003, p. 95, our translation). The meanings attributed to the concepts worked are, therefore, mediated by the actions of the teachers. The students, in this perspective, become subjects in the pedagogical activity as they actively and intentionally participate in the concept appropriation process (Basso, 1994, our translation). Therefore, this demonstrates the importance of pedagogical intentionality in conjunction with mathematical knowledge, attributed by teachers when developing the learning triggering situations presented in this paper.

When we do not work in this way, approaching the conceptual nexus of the concept to make possible for the development and learning processes to be set in motion, the tendency is to reproduce techniques and procedures. In this way, we approach to everyday knowledge more than scientific knowledge and distance ourselves from the main objective of school education.

13.5 Final Considerations

Based on the assumptions of the Teaching-Orienteering Activity, we highlight the importance of the historical–logical process in the organization of mathematics teaching (de Moura, 1996, 2010), for it is from its knowledge that we understand the necessities that triggered the elaboration of the concepts. With this understanding, we can produce teaching activities that evoke concepts to be taught through needs similar to the historical process by which such a concept was produced. This is the essence of the methodological perspective put forward by the Teaching-Orienteering Activity (TOA), which goes much beyond the idea of simple movement; neither fits the ideas commonly present in the common sense of activity understood by mere actions disconnected from the essence of the concept.

For this reason, the act of reflecting on what we teach is an essential action in pedagogical activity. Because, according to de Moura (2004):

> Subjects who deal with the concept as a tool need to have access and means that lead them to a very precise understanding of their object, because they need to give meaning to what they teach so that their students can make sense of what they are told is important to learn. It seems to us is to be the first and main problem that we must address in teacher formation. This is a professional that we could call a creator of meaning for what is taught and his main tool is the word (de Moura, 2004, p. 258—our translation).

For this, it is necessary to understand the human need to create such knowledge so that it can be taught in a meaningful and non-mechanized way. Thus, because we do not reflect and analyze knowledge as a social object, that mechanized teaching may occur. For Davydov (1982), reflective activity:

> […] allows modifying the ideal images, the "projects" of things, without modifying the things themselves until that moment. This modification of the "project" of the thing, based on the experience of its practical transformations, involves the kind of subjective activity of the human that in Philosophy is usually called thinking. Thinking means inventing, building in the mind the idealized project (corresponding to the purpose of the activity, its idea) of the real object that will result from the supposed work process. Thinking means change according to the ideal project and the idealized scheme of the activity, transforming the initial image of the work's object into another idealized object. (p. 294—our translation)

This means that reflection is needed so that the initial image of an object can be transformed into another idealized object corresponding to the first one. For the purpose we present in this paper, the initial image of the objects of knowledge "measurement of time" and "area" can, through reflection and analysis, be transformed into idealized objects that correspond to the purposes of the teaching activity to be elaborated.

As an example, we can mention the awareness on the part of one of the teachers, exposing her discomfort when realize, with the studies on the concept of measurement of time carried out during the educational process, that the teaching of such a concept, during the five initial years of schooling, is assimilated in a mechanized way, working only with the use of instruments as a final product. In her own words, the teacher reports:

> - Wow, we spend five years teaching to read a clock and the kids don't necessarily learn to read the time, and to get to solve problems of how many minutes I was late or early. It's very procedural.' It got like this, a nuisance, wow! There it is! On top of this story, what is a concept anyway?! (Gladcheff, 2015, p. 206—our translation).

With that, the teacher starts to reflect on: Why do we teach such a concept?—What should we teach about such a concept?—How can we teach about such a concept?

In this context, the learning triggering situations materialized through the virtual stories "Vegetable Garden" and "A problem to the curumins" were elaborated by a collaborative collective that, in our understanding, due to the functions of its participants (Basic Education teachers, pedagogical coordinators, graduate, undergraduate students, and university professors), provided a precious movement of elaboration, writing, application, re-elaboration, and rewriting. This process contributed to the elaboration and evaluation of teaching activities from different perspectives.

Considering that "men do not live in a constant creating state. He only creates for necessity, that is, to adapt to new situations, or to satisfy new necessities" (Vázquez, 2011, p. 269, our translation), the challenges were established as an integral and essential part of the process, since, according to the aforementioned author, without the need to create, man only repeats what exists. However, "creating is for him the first and most vital human need, for only creating, transforming the world, men […] makes a human world and makes himself." (Vázquez, 2011, p. 269, our translation).

With this way of acting, it is possible to evidence the planning and materialization of the teaching actions of teacher, marked by the intentionality that indicates the level of awareness of their role in the teaching process and in the social group to which they belong, such as the "process of awareness of its value as a subject of a community that seeks to solve problems." This, according to de Moura (2000), "is what will allow you to take on the desire to change and to form autonomously in consonance with collective objectives" (de Moura, 2000, p. 45—our translation).

Thus, the coming and going, the integration of theory with practice posed by the dialectical movement of re-elaboration of the proposals in activity, as well as the return to rethink the theoretical construct that, guided, allow the materialization of the process of reflection, analysis, and synthesis present in the cycle of action and transformation of reality is materialized as a human activity.

In this sense, we seek to present, in this article, contributions to the discussion on teaching area and measurement of time in the early years of basic school, in order to show, in the case of the area, its surface quantifying character, to the detriment of teaching based on formulas; and, in the case of time measurement, the possibility of "non-procedural" teaching that aims not only the development of time reading and calendar filling techniques.

Based on what we have presented, we highlight that this is a proposal that enable the appropriation of more elaborate knowledge. However, we emphasize that the difficulty for such a proposal to be implemented in schools is initially linked to teacher training. The possibility of working with this approach occurs when the teacher expands their knowledge, and this can happen through participation in continuing educational processes and/or in initial training, as long as there are educators who emphasize the theoretical–methodological proposal of the Teaching-Orienteering Activity not as a script to be followed, but as a pedagogical possibility of creation with the purpose of human development.

Whether with teachers in the early years of schooling or not, the importance of studies on the logical–historical movement of mathematical concepts and the theoretical–methodological principles, which underlie the Teaching-Orienteering Activity, must be present in the educational processes. This is an action of teacher educators, which has been developed by members of GEPAPe (Group of Studies and Research on Pedagogical Activity), in ongoing and initial educational processes.

As one of the educators participating in such a movement, Gladcheff (2015) emphasizes the partnership between university and basic school education in carrying out training activities by evidencing that:

> Teachers, by directly experiencing the difficulties of everyday school life, provide a reference to academic researchers who, often, do not participate in the important articulation between theory and pedagogical practice. This partnership, however, must include the school as a place of training and learning, both for teachers and for researchers, with the main focus on the social meaning of school education. The way we build the research action, approaching the collaborative work, make possible the (trans) formation of all its members, as it provide to the group the knowledge they carry with them and this is only possible because there are several people in interaction that have a common objective: to intervene in the educational process with a view to the maximum humanization of the individuals who participate in it. (Gladcheff, 2015, p. 248—our translation).

It is noteworthy that public policies, with official proposals such as the Education Observatory[3] program that made possible to carry out the network project mentioned in this text, and which, recently, were no longer implemented in our country, need to be resumed with the intention of returning to the agenda as a state policy and not a government one. This so that basic school education and the university have, indeed, not only possibilities, but the materialization of incentives for integration.

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[3] The Education Observatory Program, the result of a partnership between Capes (Coordination for the Improvement of Higher Education Personnel), INEP (National Institute of Educational Studies and Research Anísio Teixeira), and SECADI (Secretary of Continuing Education, Literacy, Diversity and Inclusion), was implemented in Brazil in 2006 and was interrupted by the last government administration. It had the purpose of promoting studies and research in education, aiming, mainly, to provide the articulation between postgraduate courses, undergraduate, and schools of Basic Education, as well as to stimulate academic production and the formation of postgraduate resources, at the master's and doctoral level.

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Chapter 14
Collective Reasoning and the Use of Learning Models for Relationships Between Quantities, as Suggested by the El'konin–Davydov Curriculum

Helena Eriksson

Abstract This chapter focuses on learning activity that was suggested by El'konin and Davydov as a framework for teachers to design opportunities for students to emerge collective mathematical reasoning. The data for the chapter were taken from a 3-year longitudinal research project in which learning activity was used to design teaching, tasks and learning models in a multicultural and multilingual Swedish primary school. As a result of using this framework, a situation was highlighted whereby the learning models specific to learning activity had been constructed by second-language students aged 6 and 7 years as they emerged collective mathematical reasoning that focused on comparing two different quantities. The contribution to previous research concerning how learning models can be used in such student groups for collective reasoning about relationships between quantities is twofold; that is, (1) the use of the learning models can help students and teachers to return to the problem they have jointly identified in the collective reasoning and (2) the use of the learning models can help students who struggle with different linguistic dilemmas to jointly reflect on both their own and others' arguments.

14.1 Introduction

Teaching that aims to promote mathematical communication, reasoning and argumentation capabilities in whole-class teaching has challenged both teachers and researchers (Planas & Valero, 2016; Radford & Barwell, 2016; Ryve et al., 2013; Skolforskningsinstitutet, 2017). At the same time, studies have shown that young children are able to conduct and follow mathematical reasoning as they develop logical chains of arguments, including communicating about mathematical properties, for example, in free outdoor play (Sumpter, 2016; Sumpter & Hedefalk, 2015)

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and in role play (Van Oers, 2018). Research on mathematical reasoning and argumentation has received increasing attention in recent decades (Lerman, 2006; Planas & Valero, 2016; Ryve et al., 2013). Specifically, interest in research on collective activities that consider the needs of individual students has increased (Fleer & Van Oers, 2018; Moxhay, 2008). As one example, arguments that include mathematical properties should not only be provided by the teacher and consequently precede students' argumentative actions; more importantly, the arguments should also be provided by the students to enable them to progressively develop more qualified reasoning (Van Oers, 2018).

In Sweden, education that aims to promote these capabilities has received more attention due to a changed social structure (Bunar, 2015). The structure that has been developed there is similar to that in other countries where student groups are often both culturally and linguistically heterogeneous. Teaching in such student groups has to be staged without shared linguistic and cultural experiences (Adler, 2001, 2019; Barwell et al., 2016; Leonard et al., 2010; Norén, 2015; Norén & Andersson, 2016; Radford & Barwell, 2016). Groups of students who lack such shared experiences are deemed to have limited opportunities to develop joint conversations if they rely solely on verbal language and what can be assumed to be shared everyday experiences (Adler, 2019; Leonard et al., 2010). Studies have shown that the transparency of content that may be necessary in multilingual environments is often lacking and that research on how such transparency should be manifested is limited (Adler, 2001, 2019; Barwell et al., 2016; Moxhay, 2008; Norén, 2015). In such student groups, collective mathematical reasoning has to be developed through resources other than verbal language alone. Opportunities for developing mathematical abilities such as communication, reasoning and argumentation in such student groups therefore depend on the use of tools other than verbal language alone (Radford & Barwell, 2016).

In learning activity, as suggested by the El'konin–Davydov curriculum (henceforth ED curriculum), teaching focuses on students' agency and their opportunities to participate in joint activities (Zuckerman, 2004) independent of their abilities or struggling learning situations (Moxhay, 2008). In the ED curriculum, Davydov and his colleagues argued that the starting point for mathematics education involves a reflection on what is conceptually fundamental to mathematics (Coles, 2021; Venenciano & Dougherty, 2014; Venenciano et al., 2021). The curriculum focuses on interrelations between mathematical concepts, and particularly on the relationships between numbers, which is seen as the most fundamental concept (Venenciano & Dougherty, 2014; Venenciano et al., 2021). The focus on students' exploration of relationships between concepts is one of the most notable aspects of the curriculum. The tasks and the specific tools (i.e. learning models) suggested for learning activity aim to motivate and engage young students in a crucial way to develop joint activities (Davydov, 2008; see also Coles, 2021; Venenciano & Heck, 2016).

The purpose of this chapter is to provide an example of how learning activity (Davydov, 2008) can help students to participate in mathematics teaching that aims to enable collective mathematical reasoning. For this purpose, a Swedish project exploring learning activity, involving teachers and multilingual students in grade 1, is used. The specific research question is as follows: What are the implications of the use of specific leaning models when it comes to enabling students' engagement in collective mathematical reasoning about relationships between quantities?

14.1.1 Learning Activity Suggested by Davydov

The ED curriculum for early mathematics follows the theoretical framework of learning activity based on Vygotsky's the cultural–historical school (Davydov, 1990, 2008). As learning activity follows Vygotsky's ideas about theoretical knowledge, subject-specific tools (learning models) developed through actions in content-rich collaborative activities become important and even necessary (Morris, 2000; Schmittau, 2003, 2011; Zuckerman, 2004, 2012, 2022). These learning models enable students to develop the capability to create and act in the world in a more qualified and independent way (Davydov, 2008; Davydov & Rubstov, 2018; Repkin, 2003). All students' voices are important for creating joint actions and learning models on theoretical knowledge objects (Bakhtin, 1981; Davydov, 2008). Further on, to create, achieve and establish a learning activity, theoretical work needs to be carried out together by students in relation to a specific problem situation (Eriksson & Polotskaia, 2017). For theoretical work, learning models can be used to explore concrete examples of theoretical knowledge (Davydov & Rubstov, 2018; Repkin, 2003). This principle is what is described as ascending from the abstract to the concrete (Davydov, 1990, 2008). Mason (2018) highlighted this principle with examples of young students developing and enhancing number sense in early school years by exploring relationships between quantities through measurement tasks (Schmittau, 2004) and using algebraic expressions in learning models for the same purpose (Dougherty, 2008). Both the exploration of relationships and the use of models, which sometimes contain algebraic expressions, for these explorations are advocated in the ED programme (Dougherty, 2008; Mason, 2018; Schmittau, 2004, 2011; Venenciano & Heck, 2016; Zuckerman, 2004, 2012). The tasks within this curriculum are often based on measurement, whereby students are invited to work with quantities represented as lengths, weights, areas, volumes (continuous quantities) and numbers of physical things (discrete quantities). Tools suggested by the ED programme that can be used to construct learning models include algebraic and numerical symbols, geometrical figures, algebraic expressions, oral and written language as well as gestures (Zuckerman, 2004).

To help students develop learning activity as a joint mathematical situation, it is important that they are given the opportunity to analyse the mathematical content that is the focus of the situation. This analysis can be done with the help of learning models. It is important in this analysis that the students themselves, together with the teacher, identify a possible problem that they can reflect on and solve together. The process of identification and reflection should be organised in a way that allows students to also take the perspectives of other students (Zuckerman, 2004). One way to organise this is to design tasks whereby students are invited to reflect on hypothetical situations in which fictitious students have solved problems in different ways (Eriksson & Polotskaia, 2017; Zuckerman, 2012). When teaching young students, the learning activity should therefore (1) encourage the students to independently search for ways to solve a new problem instead of presenting a ready-made pattern for solving; (2) rouse and maintain the students' initiatives of questions, assumptions and observations; and (3) create conditions that will motivate the children and enable them to interact with their classmates so that they understand and develop each other's suggestions about the ways to solve a new problem (Zuckerman, 2022).

Learning activity that offers joint mathematical situations and include work with learning models, as proposed in the ED curriculum, can, through the opportunities given by the models for reflection, make the transmission of theoretical knowledge more transparent (Davydov, 1990, 2008; Moxhay, 2008). This is what Storch (2017) argued is necessary and important for students in multilingual classrooms. Previous studies in multilingual classrooms have suggested that the learning models proposed by the ED curriculum can help students reflect on theoretical knowledge, regardless of their struggle with language difficulties (Eriksson & Eriksson, 2021). For example, in this study, the students identify the relationship between addition and subtraction by reflecting on the fact that the statements $C = A + B$ are equal to $C = B + A$ but that $C = A–B$ is not equal to $C = B–A$. The students consider that the commutative law does not apply to subtraction by using the length models suggested by the ED programme (Eriksson & Eriksson, 2021).

14.1.2 Collective Mathematical Reasoning

Mathematical reasoning can be described in different ways and perceived as individual or collective (Sumpter, 2016). The focus of this chapter is on collective reasoning, which in turn can be discussed in many different ways depending on, among other things, basic theoretical assumptions. Researchers and teachers struggle to understand many of the complex relationships between students and teachers in collective reasoning, the engagement of participants in whole-class collective reasoning and how mathematical properties are explicitly conveyed in such reasoning situations (Ryve et al., 2013). One way in which studies have focused on collective mathematical reasoning is by discussing how to achieve it (Davydov, 2008; Radford & Roth, 2011; Webb et al., 2014; Zuckerman, 2022). This was the focus of, for example, Radford and Roth (2011), who argued that students need to be invited into spaces for collective action where the role of the teacher and how students are involved in the activities are particularly important. These researchers suggested that students and teachers are assigned different roles in classroom interactions. 'Togethering' refers to the role of the teacher and is used to emphasise the teacher's importance in capturing a collaborative activity that also involves the teacher and aims to realise the commonly motivated goal of the activity. Using Webb and colleagues' (2014) research, the students' roles can be described in two ways; that is, they explain their own mathematical ideas and they engage with their classmates' ideas, explanations and arguments (Webb, et al., 2014). As described earlier, learning activity, as suggested by Davydov (2008), also focuses on joint activities and on students' opportunity to be agents in their own activities. The concepts of learning activity, learning models and learning tasks are intended to help teachers support collective reasoning. Consequently, there are several aspects of what happens in the classroom, such as mediating tools and the teacher's responsibility to focus on mathematical properties and interactional features, that are important for collective mathematical reasoning (Davydov, 2008; Repkin, 2003; see also Ryve et al., 2013).

It is important for both teachers and researchers to find a structure to describe and analyse collective reasoning. For this purpose, mathematical reasoning can be considered as a line of thought in which the arguments are connected through mathematical properties (Lithner, 2008, 2017). Collective mathematical reasoning can be carried out as collective joint activities that aim at meaning-making through the arguments of different participants that contain mathematical properties (Sumpter & Hedefalk, 2015, 2018). In such examples of collective mathematical reasoning, the arguments are provided by all participants, including the teacher and the students (Sumpter, 2016). Mathematical reasoning has a structure that can be explained as follows: (1) a (sub) task is explored (TS), (2) a strategy choice is made (SC), (3) the strategy is implemented (SI) and (4) a conclusion is proposed (C) (Lithner, 2008, 2017). This structure consists of arguments that are linked to each other by mathematical properties. Research has identified four types of argument included in the different parts of mathematical reasoning (Eriksson & Sumpter, 2021; Hedefalk & Sumpter, 2017; Lithner, 2008; Sumpter & Hedefalk, 2018). These different types answer different questions concerning how a problem can be solved, and the identified arguments are linked to the introduction of a task situation (Eriksson & Sumpter, 2021). These arguments answer the question, 'which problem should be solved?'. Next, predictive arguments are linked to the choice of strategy, answering the question, 'why will the strategy solve the task?', and verifying arguments are linked to the strategy implementation, answering the question, 'why did the strategy solve the task?' (Bergqvist & Lithner, 2012). Finally, evaluative arguments are linked to the conclusion, giving an answer to the question, 'how does the conclusion answer the question for the investigated (sub) task?' (Hedefalk & Sumpter, 2017; Sumpter & Hedefalk, 2018).

14.2 Methods for the Research

The data for this chapter were produced in a three-year longitudinal research project conducted in Sweden from 2014 to 2017. The aim of the project was to use the ED curriculum as design for teaching one math lesson a week. The other lessons followed a traditional Swedish teaching approach. The project took place in a multilingual school consisting of preschool and primary school grades 1–6. This school is in an area considered one of the most segregated areas of Sweden. From this project, a whole-class lesson was selected from the first term of grade 1. The students were aged 6 or 7 years, and all were second-language learners of Swedish. About 20 to 25 mother tongues were spoken at the school, including five different languages in this class. There were twelve girls and eleven boys in the class. The pupils' capabilities in mathematics were varied, according to national and local mathematics screening tests. The students were not familiar with mathematical concepts either in Swedish or in another language.

The research project followed the principles of design and action research (see Zuckerman, 2022). It was carried out by a working group consisting of the author of this chapter and six professionally qualified teachers with different levels of teaching experience. The group worked together to design the lessons according to the principles of learning activity and in such a way that the lessons were planned, implemented, analysed and revised in an iterative and cyclical process. For more information about the research project, see Eriksson (2021).

The selected lesson was documented on video and transcribed with students' and teachers' actions focusing on the task in Fig. 14.1. In the transcript, the students' names are fictitious. All students and their guardians provided written consent to participate in the project, according to the Swedish Research Council's standards (Vetenskapsrådet, 2017). The consent and other information about the project were described to the parents at meetings by native language teachers and interpreters.

The task for the lesson was inspired by the ED curriculum and focused on the central ideas of learning activity to use and develop learning models in joint activities.

A photo	A picture
A length model and letter symbols	Numerical examples

Fig. 14.1 Four fields to be constructed by the students when exploring different models for relationships between quantities

A task should allow the students to reflect on the theoretical content embedded in the problem that they have identified (Davydov, 2008). The mathematical aspect in focus concerned the relationships between numbers and different ways of representing these relationships. As usual, when using the ED curriculum, the task was based on measurement, where the students had to work with quantities represented as volumes, length models and symbols. The task was conducted as a joint activity by the whole class. The first part of the task, presented here, focused on mathematical inequality. The students were asked to construct four fields on a paper with help of a ruler and a pencil (Fig. 14.1).

The problem the students were tasked with was to identify inequalities and how to represent them so that the differences between two quantities could be explored. However, as mentioned earlier with regard to learning activity, the problem was not provided to the students but instead identified by them. After constructing the four fields on the paper, a discussion on mathematical inequality was initiated, which the students were expected to continue with their friends or families at home. The first problem was then to work out how they could discuss these inequalities outside the classroom. The problem was solved when the students summarised their joint conversation with the need to represent the mathematical inequality. For this representation, different tools, or what are called learning models in learning activity, should be used.

The first model of inequality consisted of two equal-sized jars containing two different volumes of water. The students were asked to reflect on what was different between the two jars and which of these differences that was important from a mathematical point of view. Depending on the students' suggestions, they were challenged with questions that led to the four models presented in Fig. 14.1, namely (1) a photo, (2) a picture, (3) symbols and a length model and (4) invented arithmetic examples. In the first field, the teacher had prepared a photo, which the students pasted into it. In the second field, the students were expected to draw a picture showing an inequality. In the third field, on the teacher's initiative, a length model specific to the ED curriculum was constructed (see Fig. 14.2); there, the students were asked to propose an algebraic symbol (i.e. a letter) for each volume so that these volumes would be easier to reflect on. The work of one of the students in the first three fields is reconstructed in Fig. 14.2, which shows the different volumes of two jars, labelled A and B. The work in the fourth field, where the students had to provide numerical examples of inequalities, is presented in the following section.

14.3 Collective Reasoning about the Relationships Between Quantities

This section provides an analysis of the multilingual students' mathematical reasoning when working with learning models to reflect on the relationships between quantities. The situation used for the analysis refers to the fourth field of the task

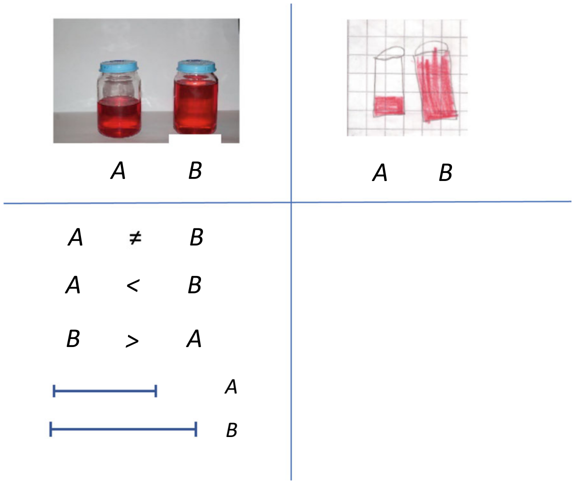

Fig. 14.2 Reconstruction of a student's work in the first three fields

presented in Fig. 14.1, when the students were asked to provide numerical examples for the inequalities they had discussed. The analysis began with the transcripts being structured according to the structure of the mathematical reasoning suggested by Lithner (2008). This structure is presented in column three of Table 14.1. The second step in the analysis was to identify the different types of arguments used in these reasoning structures according to the different types of arguments proposed by Eriksson and Sumpter (2021). These arguments are also presented in column three of Table 14.1. In the third step, the mathematical properties (e.g. content, concepts and transformations) that anchored the arguments were identified and presented in the fourth column of Table 14.1.

Table 14.1 implicates that when the students were working jointly on the fourth field in the task in Fig. 14.1, the learning models could affect the students' collective reasoning in two different ways. First, the learning models helped the students and the teacher to identify that, and return to, the problem they were solving was about determining different volumes of water in the jars so that the expression $A < B$ would be true. Second, the learning models helped the students to use each other's arguments to identify that the algebraic expression and the length models were models of the volume of water in the two jars. In this joint work (i.e. potential learning activity), the teacher collaborated with the students by asking questions and orchestrating the collective reasoning. In Table 14.1, the details for the first way that the learning models affected the students' collective reasoning are highlighted in bold, and those for the second are in italics.

Table 14.1 The problem situation when the activity was to find numerical examples for *A* and *B*

Time code	Data	Reasoning structure (type of argument)	Mathematical properties
20:32	Abel: **Do you mean we can suggest any numbers?**	A new TS was identified	$A = n; B = m$ $n = \{\infty\}$ $m = \{\infty\}$
20:34	Bea: For the water or the letters? [**Holds up her book and points at the jars and the letters.**]	SC predictive argument	$A, B =$ what? $A = n; B = m$
20:34	Dick: **Suggest what?**	SI	
20:40	Teacher: **Wait.** First, we have to think about what the difference is between the letters and the water. **[Points at the letters and the water in the jars.**]	C evaluative question reinforcing TS using the letters	Representations $A \neq$ the water $B \neq$ the water
20:45	Students: 4 and 5	SC	
20:45	Teacher: **Wait.** Do the letter symbolise the water or the volumes [points at the photo of the jars in the first field of the task] and what do the letters symbolise? [**Points at the letters below the jars**] And also, these length models, what do they symbolise? [**Points at the length models in the third field in the task on the board.**]	C evaluating question reinforcing TS using the letters, and the length models	What representations?
21:20	Students: 4 and 5	SC	$A = 4; B = 5$
21:20	Eric: 20	SC	A *or* $B = 20$
21:25	Teacher: **Stop.** No more numerical suggestions. What did we say about the letters? What do they represent or symbolise? [**Points at the letters below the jars.**]	C evaluating question reinforcing TS using the letters	A concluding question regarding *A* and *B*
21:35	Dick: *The jars? [Questions and goes to the board and points at the jars.]*	SC	*A* and *B* are representing an object
21:35	Eric: *Lengths? [Questions and holds up his book and points at the length model.]*	SC	*A* and *B* are representing a model
21:35	Fred: *Do they symbolise the jars?*	Reinforcing TS identifying	What do *A* and *B* symbolise?
21:39	Dick: *Yes*	SI	
21:40	Teacher: Do they?	Reinforcing TS	

(continued)

Table 14.1 (continued)

Time code	Data	Reasoning structure (type of argument)	Mathematical properties
21:43	Abel: *The alphabet. [Points at the wall with the alphabet.]*	SI verifying by pointing at a model	A and B are letters
21:43	Dick: *Letters*	SI	Representations
21:44	Teacher: What do the letters symbolise in this task? What do they represent here? [**Points at the letter symbols.**]	C evaluating question reinforcing TS	$A =$ $B =$
21:44	Dick: *The jars… [Goes to the board and points at the jars in the photo.]*	SI verifying using a model	Representations
21:45–22:10	Teacher: Do they? [The discussion goes on for a while. It ends up in a disciplined discussion by the teacher.]		
22:10	Teacher: Listen. What did we talk about when we decided on what letters to use? [**Points at the letters.**]	SI back to TS Verified argument by pointing at the letters	$A =$ $B =$
22:20	Dick: *The jars. [Points at the picture of the jars in his book.]*		
22:22	Glenn: *No, actually… it is the water that is different…*	C evaluating argument	Implicit that the volume of the water is not equal Implicit that $A \neq B$
22:24	Fred: *One is small and one big. [Points at the length models.]*	SC	$n < m$
22:24	Dick: *A is small. B is large*	SI verifying using the models consisting of algebraic symbols	$A < B$
22:25	Glenn: *A is before B. [Points at the wall with the alphabet.]*	SI	Meaning the alphabet: A, B, C
22:27	Teacher: Yes. Listen carefully. What do the letters symbolise? [**Points at the letters.**] Anyone else?	Reinforcing TS pointing at the letters	
22:35	Dick: The water…	SI	
22:35	Teacher: …or, what about the water?	Reinforcing TS	

(continued)

Table 14.1 (continued)

Time code	Data	Reasoning structure (type of argument)	Mathematical properties
22:37	Fred: How much water? It's different	C	$A = n$ $B = m$
22:37	Teacher: Yes, and why did we use the letters?	Reinforcing TS	
22:39	A student: To describe	SC	
22:39	Teacher: Yes, or to symbolise. To symbolise what?	SI	
22:42	Fred: How much water…	C	
22:42	Teacher: Yes, and in Swedish, we call that the volume. Have you heard that word before?	C evaluating argument	A and B are representing the volume
23:05	Dick: Yes, a bottle	SI	
23:05	Teacher: Let's go on. We should suggest numbers for the volumes. How do we do that? Can we suggest any numbers?	Reinforcing the first TS of this situation	$A = n; B = m$ $n = m = \{\infty\}$
23:10	Eric: Yes	SC	
23:10	Fred: No, A has to be smaller than B	SI	$A < B$
23:10	Teacher: Why?	Reinforcing TS	
23:25	Fred: Look. *A is just a little water in this* [*goes to the teacher's desk and* ***points at the jar*** *with the volume labelled A*]*; therefore, A is small. Look here, this is B; it is almost full, and it its bigger.* [***Points at the other jar***.] *Therefore, A is 5 and B is 20. B is a* ***long stretch***	C evaluating arguments using a combination of tools	$A < B$
23:45	Teacher: This is important. What did Fred say? Can anyone explain?	SI: Asking for the evaluating argument again in C	

14.3.1 The Learning Models Helped the Students and the Teacher to Identify and Return to the Problem

Regarding the first way the learning models affected the students' collective reasoning, they were used to identify the problem that the students were to solve in the fourth field (bold text in Table 14.1). At the start in the situation below, Abel (at 20:32) initially identified the problem regarding what numbers the students might suggest

($n = \{\infty\}$, $m = \{\infty\}$). Abel, Bea and Dick, identified this implicitly formulated problem as, 'What should they give numerical examples of?' In the task construction, the water as well as the letters were meant to symbolise different quantities, namely different volumes, but according to the predictive question of Bea (at 20:34), the students did not discern the letters as symbols for the volume, and therefore the quantity of water. Instead of focusing on the numerical examples to explore the relationships requested in the fourth field, the teacher's evaluative argument focused on taking the students back to the argument they had first identified. Although the joint activity in the three previous fields focused on exploring what the models and symbols depicted, the work in the fourth field, which should have focused on the numerical examples for the unknown quantities, focused the same thing. The teacher focused on the models by repeatedly asking what the models were of ($A \neq$ the water, $B \neq$ the water, $A =$ volume, $B =$ volume) at least three times (at 20:40, 20:45 and 21:25) and pointed to and verbally referred to them. The students' strategy choices subsequently focused on the models and the letter symbols, identifying that they were symbols for the quantity of water. The teacher verified their arguments several more times by taking the students back to their identified problem (at 21:44, 22:10, 22:27, 22:37 and 22:39), sometimes pointing at the different models and sometimes asking questions. Fred's conclusion when evaluating the task situation was linked to the teacher's persistently repeated questions about the symbols.

14.3.2 The Learning Models Helped the Students to Use Each Other's Arguments

Regarding the second way, one example of the use of the learning models concerned the opportunities for the students to argue with the help of both their own and others' arguments (italic text in Table 14.1). Dick, Fred and Glenn reflected on identifying the problem to be solved (at 21:35–21:39 and 22:20–22:25 and with the teacher at 21:43–22:10) in Table 14.1. They pointed at the jars with the water, the amount of water, the letter symbols and the length models to jointly compare the quantities of the volumes. Fred evaluated $A < B$ (at 23:25) by using the volume, the letter symbols, the length models and a numerical example. The students carried on these reflections by using oral language and pointing to models on the board. In this problem situation, the students could not only explore the relationships between the quantities but also the letters as unknown quantities and as a model for the relationships.

When the students at the beginning of the situation in Table 14.1 were asked to determine numerical examples for A and B, they did not know what they should give examples of. At the end of the situation Fred evaluated, with help of the learning models, that A and B were connected to the volume, thus the quantity of water (at 23:25). All students then were able to suggest individual numerical examples explicitly grounded in the learning model that consisted of the expressions with algebraic symbols, see Figs. 14.3 and 14.4. In this fourth field, the students could

Fig. 14.3 Mohammed's work in the fourth field in the task

A	$<$	B
5	$<$	20
5	$\neq$	20
4	$<$	5
17	$<$	18

Fig. 14.4 Nadiro's work in the fourth field in the task

A	$<$	B
5	$<$	20
5	$\neq$	20
4	$<$	100
10	$<$	10 + 2
10	$<$	17 + 3 + 5
5 + 5	$<$	100

do their individual explorations. Nadiro, for example, presents three operations with addition, see Fig. 14.4.

Table 14.1 shows the work in the fourth field when the teacher asked the students to provide numerical examples for A and B that would make the inequality $A < B$ true.

14.4 Conclusion

The implications of using the specific learning models shown in the collective reasoning are above all the assistance they offered the students and the teacher to identify and return to the problem, as well as the support they provided for the students to use and develop each other's arguments. The learning models helped both students to understand, and the teacher to highlight, that it was the differences between the two unknown volumes that they should reflect upon. In strategy choices and implementations, students reflected on whether it was the water or the volume that was of interest. The algebraic letters were thus used to focus on the quantities both in terms of the physical volume, the length models, and finally also the suggestions for numerical examples. The students and the teacher then pointed at and verbally referred to the learning models on the board. In that manner, the three tools—algebraic symbols, the length models and the physical volumes—in the learning models helped the students to use each other's arguments, and the teacher to challenge the students by mentioning the letters and pointing towards the models on the board. The learning models helped in terms of enabling students' engagement in collective mathematical reasoning when analysing the relationships between quantities.

14.5 Discussion

The main results of this chapter indicate that young second-language learners, through tool-mediated joint reflective actions, as suggested in the ED curriculum, succeeded in developing collective mathematical reasoning. The suggested learning models offered the students and the teacher to identify and return to the problem. In addition, they provided support for the students to use and develop each other's arguments.

The students in this study initially had difficulty understanding the meaning of the algebraic symbols. Through the joint activity, they explored these symbols and simultaneously analysed the relationships between different quantities. These relationships are examples of structures in arithmetic that are challenging for not only second-language learners but also many other young students (Adler, 2019; Barwell et al., 2016). However, these types of joint learning models succeed in mediating mathematical concepts and visualising mathematical content, which, for example, Adler (2001) and Storch (2017) argued are important features of mathematics teaching in multilingual classrooms. Moreover, the starting point for this chapter was that although young children can follow and exhibit mathematical reasoning (e.g. Sumpter & Hedefalk, 2015; Van Oers, 2018), researchers and teachers struggle to create situations for students to engage in (e.g. Planas & Valero, 2016; Radford & Barwell, 2016; Radford & Roth, 2011; Skolforskningsinstitutet, 2017). To make these situations possible, researchers have argued that students need to be invited into spaces for joint actions, such as learning activity (Davydov, 2008; Radford & Roth, 2011; Zuckerman, 2004). The findings in this study demonstrate that such joint spaces benefit from teachers and young students working to develop learning activity also in multilingual student groups.

Altogether, two elements of the lesson appeared to be particularly important in enabling the multilingual students to engage in collective mathematical reasoning. First, and in line with what Eriksson and Eriksson (2021) showed, students' ability to reflect on the symbols to identify relationships between quantities seems to be linked to their joint construction of learning models. The construction of learning models enabled them to use each other's arguments in their reflections. Second, these reflections seemed to be possible because the students repeatedly verified and implemented content details in line with the problem they had identified. The problem identification was done by identifying arguments, which had been investigated and defined by Eriksson and Sumpter (2021) as the first step in the structure of mathematical reasoning that developed into a learning activity. During the lesson, the different types of arguments, including the identifying arguments, were staged not only by the teachers but also the students. Therefore, these learning models seemed to be particularly important for the multilingual students to elicit their reflections on what the symbols were symbolising and the relationships between quantities. Thus, for a joint activity in which students can develop collective mathematical reasoning, learning activity can be used as a framework for designing teaching (Davydov, 1990, 2008; Radford & Roth, 2011), or, more specifically related to young students' emerging understanding of mathematics, the ED curriculum can be used as a framework for designing their teaching (see also Davydov, 1990, 2008; Schmittau, 2003, 2004, 2011; Venenciano & Heck, 2016; Zuckerman, 2004, 2022). Important to note is that in the teachers' design work using the ED curriculum, it was not possible to know or decide exactly what kind of learning models that would be developed during the lessons; rather, the models depended on the students' suggestions. The teachers in the project had to prepare for several types of arguments by the students, but over time, they became increasingly familiar with what to prepare.

Finally, to find out more about how the ED curriculum and the learning models it proposes can influence collective reasoning in multilingual classes, more models and tasks designed for learning activity need to be explored (Coles, 2021). Planas and Pimm (2023) argued for the importance of exploring the possibilities of different mediating tools, such as language and other communicative tools, in a multilingual mathematical classroom. Learning models in this context may be seen as a combination of language as well as other mediating tools. More research on the ED curriculum and the learning models may lead to changes in instruction over time, with the result that teachers and students become keen to analyse and formulate mathematical problems and content together using different mediating tools. One example of further research on learning activity is the Swedish project led by Professor Inger Eriksson 'Problem situations, learning models and teaching strategies—design of a professional support for the youngest students' understanding of the positional system' funded by the Swedish Institute for Educational Research (2023). The purpose of this collaborative design project is to create a professional framework of tasks, tools, models, and theoretical principles based on the ED curriculum to be used by teachers to improve f-3 pupils' understanding of the structure of the positional notation system regardless of base.

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Chapter 15
"Learning Models" as a Means of Materialising Algebraic Thinking in Joint Actions—Results from a Design Study in Grades 1 and 5 in Sweden

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Abstract The aim of this chapter, which draws upon data from a design research project based on Davydov's principles of learning activity, is to exemplify the functions learning models can have in visualising students' algebraic thinking when they collectively discuss algebraic expressions. The data are comprised of videotaped research lessons in grades 1 and 5, respectively. The analysis indicates that the learning models enhanced the students' joint exploration of the idea behind algebraic expressions and thus materialised their algebraic thinking in the whole-class discussions in three ways: (1) materialising an argument, (2) materialising a problem, and (3) materialising a collective memory.

15.1 Introduction

This chapter reports on a Swedish design study framed by learning activity in mathematics (Davydov, 2008). Learning activity can be seen as an educational theory developed by El'konin and Davydov in the 1950s and 1960s (Repkin, 2003a), and based on the principles of cultural historical theory (Vygotsky, 1963, 1986) and activity theory (Leontiev, 1978). The main idea behind learning activity is the development of students' theoretical thinking, that is, in this case algebraic thinking. In order to develop such thinking, Davydov (1990, p. 173) argues that the teaching has to be organised following the model of "ascending from the abstract to the concrete." In other words, the students should first be invited to collectively explore the theoretical content, the abstract, for example, the structural aspects of an algebraic expression.

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However, as the abstract is not instantly observable for the students, they need a mediating artefact, in learning activity known as a learning model, that has the power to visualise and explore the abstract (Gorbov & Chudinova, 2000). The work done by students using a learning model can therefore be described as collaborative theoretical work. This brings the concept of the theoretical content and learning model to the forefront of learning activity as an instructional approach. This chapter aims to exemplify the function of a learning model as a mediating tool for materialising algebraic thinking when students jointly discuss algebraic expressions.

In recent years and with references to Vygotsky (1987) and Bakhtin (1987), there has been a growing interest in students' participation in classroom discourse and whole-class discussions (Xu & Mesiti, 2022). This interest is particularly relevant in relation to issues such as supporting students' mathematical thinking and reasoning and their reflective capabilities when analysing other students' explanations (Cobb et al., 2001; Larsson, 2015; Nathan & Knuth, 2003; Sfard, 2008). According to Gal'perin (1969), tools and models create conditions for materialising and verbalising students' cognitive processes (Engeness, 2021; Nathan & Kim, 2009). Building and reinforcing collective tool-mediated reasoning in education can thus contribute to the development of higher-order thinking (Vygotsky, 1986). Larsson and Ryve (2012) argue that productive classroom discussions are not always easy to achieve, since discussions may tend to focus on procedural issues even when students provide valuable input that could lead to fruitful discussions (see Cobb et al., 2001). In one study, Ryve et al. (2013) noted that the quality of classroom discussions improved if students were given access to mediating tools, such as tables, rather than relying solely on verbal representations (cf., Sfard, 2008). This is supported by the works of Leung and Bolite-Frant (2015) and thus highlights the importance of mediating tools in promoting high-quality content in classroom discourse.

15.1.1 Algebraic Thinking and Early Algebra

Algebraic thinking is an important part of mathematical thinking, and in recent years, several researchers have found it beneficial to develop algebraic thinking from an early age (Blanton et al., 2015; Hodgen et al., 2018; Kaput et al., 2008; Kieran, 2018; Radford, 2018; Radford & Barwell, 2016; Warren & Cooper, 2009). Lins and Kaput (2004) argue, with reference to the Vygotskian school and Davydov, that algebraic problem-solving and algebraic thinking could be introduced as early as grade 1 (see also Cai & Knuth, 2011; Kieran et al., 2016; Schmittau, 2004, 2005; Venenciano & Dougherty, 2014). The El'konin–Davydov programme in mathematics invites the students to explore possible structures and relationships between quantities through problems based on measurement (Schmittau, 2004; Schmittau & Morris, 2004). Below, some basic principles of learning activity are outlined.

15.2 Learning Activity as Theoretical Foundation

Learning activity is both a theory with certain concepts and principles that teachers can use when planning teaching for theoretical thinking, and a theory of students' agency that promotes self-regulated learning. Or to phrase it in another way, learning activity is realised when and if the students become active members in a subject-specific learning community. Some of the core concepts of learning activity are *problem situations, learning tasks*, and *collective reflections.* As mentioned above, *learning models* also play a central role in students' engagement in a learning activity (Davydov, 2008; Repkin, 2003b).

Students can establish a learning activity if they encounter a problem situation (Repkin, 2003b) that is intriguing enough for them to act on. This requires the problem situation to be content rich and contradictory. A contradiction that is built into the problem situation can be a dilemma or apparently incompatible conditions, which require students to analyse and reflect on the situation. With the support of the teacher, students identify the challenging problem and define a learning task that they see as meaningful to solve. They explore the tools at hand and what in their prior knowledge they can use for the task. A learning activity requires that the students elaborate the learning task through tool-mediated collective reflection (Engeness, 2021; Gal'perin, 1969; Zuckerman, 2004). By drawing on each other's competences, or secondary experiences, they can exceed their current competences (Vygotsky, 1986).

15.2.1 Learning Models as Mediating Tools

Davydov (2008) argues that learning models can provide a necessary link for the development of theoretical thinking (cf., Gal'perin, 1969; Gorbov & Chudinova, 2000; Repkin, 2003b). A learning model is not to be confused with a mathematical model, rather it is a form of visualisation that, in relation to a specific learning task, can help students explore theoretical aspects. Learning models are thus seen as a special form of abstraction:

> … where the visually perceived and represented connections and relations of the material and semiotic elements reinforce the essential relations of the object. /…/ This is a unique unity of the individual and the general, where the general and essential come to the fore. (Davydov, 2008, p. 95)

For example, El'konin–Davydov's mathematical learning activity uses a line segment (represented as I——I) as a learning model from grade 1. The line segment model is intended to enable the youngest students to explore what is similar and what is different and, by extension, how a difference can be mathematically transformed into a similarity. In Fig. 15.1, there are two identical jars with the same content but different amounts of the content. The two horizontal, parallel line segments below the jars signify that the jars share identical content. The two vertical line segments to the right of the jars signify that the quantities in the two jars are not the same. Since

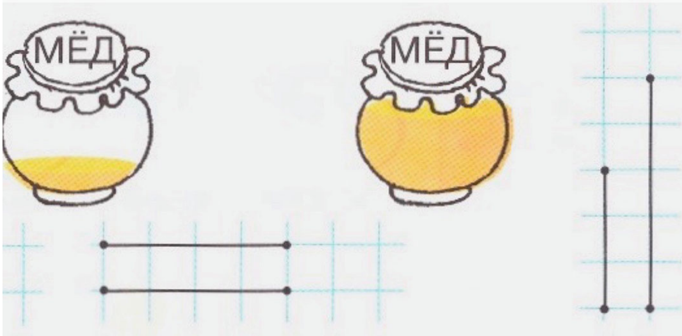

Fig. 15.1 Two jars filled with honey, and two pairs of line segments. *Note* Extract from the teaching material for grade 1, Matematikka 1 (Davydov et al., 2012, p. 19)

the quantity cannot be determined by any number, it must be understood generally in a relational way.

The quantity in the jars can be named using algebraic letter symbols. Thus, the quantity in the left jar can be labelled a and that in the right b. It is then possible for students to determine that $a < b$. Establishing a relationship between the line segments, the algebraic symbols creates a learning model that allows students to explore different general and structural aspects of an expression or a statement. This enables mathematical discussions on equality and inequality and possible transformations from different to similar as general operations (Krutetskii, 1976). Allowing students to work only with the "naked," that is, the abstract algebraic expression $a < b$ does not make sense. Admittedly, with guidance, the students can model how the expressed inequality $a < b$ could be transformed into similarities in various ways, for example: $a + c = b$; $b - c = a$; $a + d = b - d$. When discussing how the volumes represented by a and b can be made equal, the expression is given a function that is more substantive than that which the bare expression alone makes possible. If the students use the line segments as a learning model, they can transform them so that they become equal by, for example, extending the shorter line segment (a) by another line segment (c). The line segments a and c together are then equal in length to line segment b. This provides a basis for transforming the expression $a < b$ into $a + c = b$. Students can then reason about other variants of generalised processing relevant to the current situation, such as the quantities in the depicted honey jars.

According to Davydov (2008), as mentioned above, a learning model creates conditions for students' exploration of the theoretical aspects of the content (Davydov et al., 2003; cf., Eriksson, 2017, 2018). In other words, students are given the opportunity to engage with the learning task theoretically with learning models as mediating and transforming tools (Broman et al., 2022; Gorbov & Chudinova, 2000).

15.3 Method and Analysis

The data for this chapter comes from a three-year research project conducted from 2017 to 2019 and funded by the Swedish Institute for Educational Research. The project has been conducted in accordance with the ethical rules governing Swedish research, and all data are handled according to Stockholm University’s instructions. All students who participated in the project have written consent of their guardians. The overall aim of the project was to explore how teaching can be designed so that students are given opportunities to develop the capability to provide and follow algebraic arguments (cf., Eriksson et al., 2021). The study was realised in the form of a series of iteratively developed research lessons (Marton, 2015) in grades 1, 5, and 7 in three different compulsory public schools and one upper secondary school. This chapter uses data from the research lessons in grades 1 and 5.[1] In these grades, three and four iteratively adjusted research lessons were taught. The students in grade 1 were between 7 and 8 years old and in grade 5 the students were 11–12 years old. The students’ families can be described as socio-economic middle class. The classes were not level-grouped.

Each research lesson was videotaped and transcribed including statements, tone of voice, and gestures (Radford, 2010; Roth & Radford, 2011). When transcribing, student names were redacted. The analysis is based on the data from the third research lesson in grade 1 and the fourth research lesson in grade 5, since these lessons were the most developed in terms of content and the use of learning models.

15.3.1 The Analysis Process

The analysis is based on activity theory (Leontiev, 1978) in which human tool-mediated actions are seen as goal oriented to, for example, solve various problems. Thus, the analysis focuses on how students, together with the teacher, initiate and conduct discussions in relation to the learning task they defined, which tools they employ, and the ways in which students use the tools. This makes it possible to discern analytically what functions the learning models can have in a classroom situation. Particular analytical focus is placed on the actions that take the form of arguments and the function of the learning model. The analysis was carried out in two steps. First, sequences where students and teachers discussed algebraic expressions were identified in each research lesson. To be analytically relevant, the learning model needed to have a significant function. In grades 1 and 5, three sequences were distinguished, respectively. Second, these sequences were further analysed with a focus on

[1] The data and the result have been presented previously in a Swedish article: Eriksson, I., Wettergren, S., Fred, J., Nordin, A.-K., Nyman, M., & Tambour, T. (2019). Materialisering av algebraiska uttryck i helklassdiskussioner med lärandemodeller som medierande redskap i årskurs 1 och 5 [Materialisation of algebraic expressions in whole-class discussions with learning models as mediating tools in grades 1 and 5]. *Nordic Studies in Mathematics Education*, *24*(3–4), 81–106.

what could be seen as signs that the students, through the discussions, had the opportunity to reflect on structural and relational aspects of the algebraic expressions. In this second step, we focussed on whether and how learning models helped to qualify the whole-class discussions, thereby indicating algebraic thinking. The importance of planned and spontaneously occurring contradictions for the students' discussions was also identified.

15.4 Results

The results of the analysis from the two selected research lessons are presented below—first the results from grade 1 and then those from grade 5.

15.4.1 *Algebraic Discussions in Grade 1*

In the first sequence, a problem situation was presented on the board and the expression $c + a = b$ together with how two fictitious students represented the expression (Fig. 15.2). The aim was to stage a problem situation where the planned learning model would enhance the collective discussion. This was done by representing unknown quantities with pieces of paper (see Fig. 15.2) in combination with an algebraic expression.

In the second sequence of the lesson, students were asked to create representations of the expression $c + b = a$ using Cuisenaire rods. The rods, like the pieces of paper, represented unknown quantities and the aim was to visualise possible relations and structures in the expression. The students' rod constructions would then be used as learning models in the collective discussions in the third sequence of the lesson.

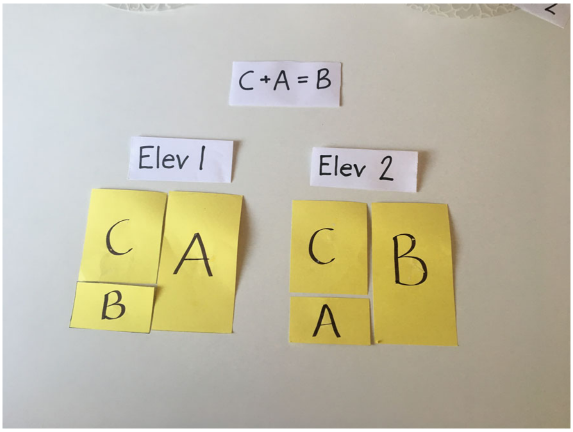

Fig. 15.2 Problem situation presented in grade 1. *Note.* Extracted from Eriksson et al. (2019)

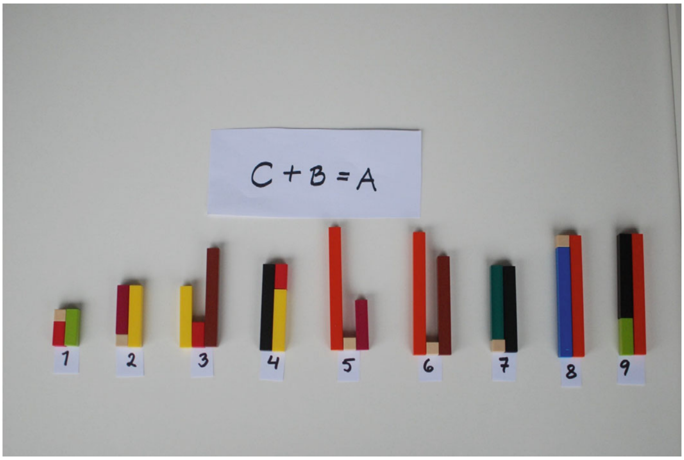

Fig. 15.3 Copies of students’ rod constructions (both correct and incorrect) based on the expression $c + b = a$. *Note*. Extracted from Eriksson et al. (2019)

15.4.1.1 “You Need Another One”

The sequence began with the teacher gathering the students around a table where she had arranged copies of the students’ rod constructions, both those that corresponded to the expression, $c + b = a$, and those that did not (Fig. 15.3). The teacher claimed: “all of these [constructions] are correct.” Several students objected to the teacher’s statement, and two of them, Adam and Elliot,[2] began to argue why not all the rod constructions could be correct.

[2] The student names used in this chapter are pseudonyms.

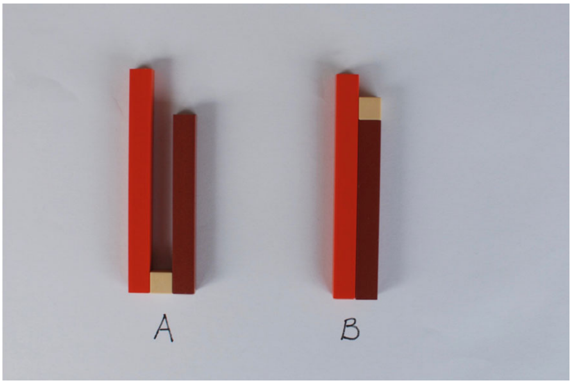

Fig. 15.4 Construction A shows the original construction and B the manipulation that Adam made in extract 2. *Note*. Extracted from Eriksson et al. (2019)

Excerpt 1

Adam:	Eh, for example, this one [points at rod construction 6 in Fig. 15.3] doesn't fit together here [points at construction 6]. So, you should do this [moves the rods so that the construction is equal to construction B in Fig. 15.4]!
Teacher:	Ok! To make it consistent [with a questioning tone]?
Adam:	No, it didn't work [moving the rods back to their original locations]!
Teacher:	What did you do? What's not working [remodels Adam's transfer]? What's not working here?
Adam:	That… You need another one [points to where it looks like one is missing if they are to be equal].
Elliot:	They've cut away a little [shows by running his finger over the rod that is "too long"].
Teacher:	Ok. Why do we need another piece? /…/
Elliot:	Because otherwise they won't have the same length.

It was only after Adam and Elliot had rearranged the rods that they realised the real problem with the rod construction was that the rods in construction 6 (Fig. 15.3) did not represent an equality. In the ensuing discussion, arguments about the equivalences shown in each rod construction were tested and reconsidered (Fig. 15.4). The arguments emphasised that a piece of rod needed to be either added or removed if the construction were to show an equality.

Adam and Elliot built on each other's ideas by experimenting with the rod constructions. The arguments were a combination of the students' verbal expressions and their experiments with the rods when testing what happened when the rods were moved. The rods visualised and fixed what could constitute a problem and thus the arguments were materialised for the students. The learning model provided the conditions for the collective theoretical work that led to the more qualified discussion that follows in the next extract.

15.4.1.2 "Several Become One Letter"

During the ongoing discussion on the meaning of equivalence, another student, Embla, pointed out that it was not enough to discuss the rod constructions individually, but that they also had to consider the written expression, $c + b = a$. She also linked to the work that had been done together in the initial sequence of the lesson and which remained written on the board (see Fig. 15.2):

Excerpt 2

Embla:	I guessed… Maybe there [points at the board where the work from sequence 1 remained] [inaudible]. One for each letter [one piece of paper or one rod for each letter].
Teacher:	One for each letter?
Embla:	Here [points at a rod construction where three rods together form an equality in accordance with the expression], but here you need [points at a construction that does not show an equality] more to make them the same length [points again at the board] where … the same length [points at the construction in Fig. 15.5], because then it will be right. More are needed here. Then you have to … Several become one letter. Here it becomes one letter, and it is not possible. There [points again at the board], oh, yes.

Embla pointed out that the rod construction (Fig. 15.5) did not match the expression as more rods were needed to create an equality. She supported her arguments from another rod construction that formed an equality, in combination with the previous learning model on the board. She also linked the rod construction to the expression and emphasised that the number of rods must correspond to the number

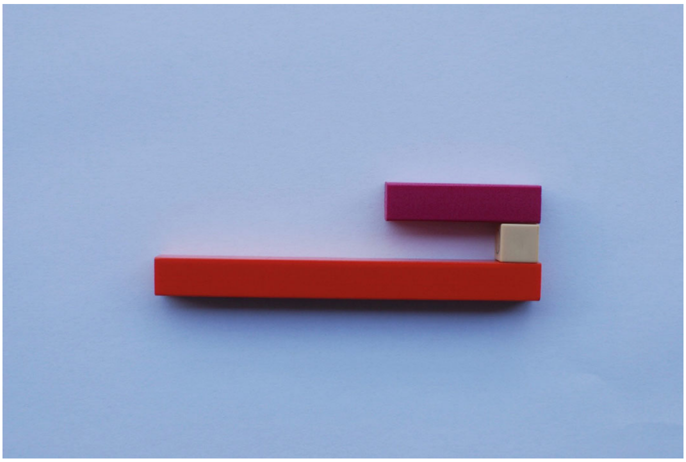

Fig. 15.5 Detail: rod construction number 5 from Fig. 15.3. *Note*. Extracted from Eriksson et al. (2019)

of variables in the expression. The learning model thus enabled Embla to further develop the work that Elliot and Adam had started. It was when Embla pointed to the pieces of paper and simultaneously manipulated the rods that the arguments were further qualified. Thus, structures in algebraic expressions as well as relationships between the variables in the expressions were materialised.

15.4.1.3 "Then There are Two c's"

At the end of the lesson, the teacher returned to another construction where Ines had used two yellow rods to make the rows of rods equal (Fig. 15.6). The teacher tried to make the students realise that the expression also needed to be adjusted to make the rod construction and the expression match.

Excerpt 3

Teacher:	But if we still have these yellow ones [Fig. 15.6]. How do we write [the algebraic expression] that? /…/ Because you say they must be different sizes [points at and touches one yellow]. How would this expression be written [moves the note with the written expression and holds it up]?
Ines:	Should the rods be equally long?
Teacher:	Yes. But if we were to write it then [points at the written expression]?
Ines:	Then …
Teacher:	Which is this [points at orange]?
Ines:	…it's… *a*!

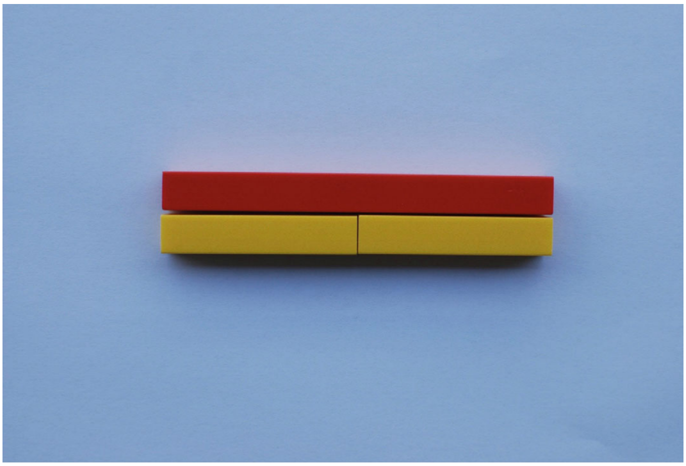

Fig. 15.6 Ines's manipulated version of one of the rod constructions from Fig. 15.3. *Note.* Extracted from Eriksson et al. (2019)

Teacher: *a*.

Ines: … and… Then it’s two *c*’s!

Teacher: Then it’s two *c*’s instead.

Elliot: *c* plus *c* becomes *a*.

Here, the teacher utilised Ines’s construction (Fig. 15.6) that had been created during the joint work and tried to use it to nudge the discussion forward by showing the contradiction that two different variables, *c* and *b*, were represented by two identical rods. In the discussion, the teacher drew students’ attention to the expression, the rods, and the relationship between them. The collective discussion finally led Ines and Elliot to recognise that the expression also needed to be changed to $c + c = a$. This can be understood to mean that the students recognised the generality of rod construction and thus chose to make the corresponding change in the expression. Students identified that the expression had the structure of “the sum of two numbers is equal to a third number,” and represented this structure by building constructions 1, 2, 4, 7, 8, and 9 as shown in Fig. 15.3. In the discussion, an opportunity was created to materialise the arguments by using the copies of the students’ constructions. In this work, the teacher and the students used the learning model to go back to ideas that had arisen during the previous theoretical work. The learning model thus enabled the modelling of the problem to become not only instantaneous but to take on the function of a collective memory.

15.4.2 Algebraic Discussions in Grade 5

In the first sequence in grade 5, the teacher introduced the learning model by presenting a picture of a line segment divided into two parts. The whole *a* and the parts *b* and *c* were illustrated with arcs on the smartboard (Fig. 15.7).

Below the picture was the question: “Which of the following expressions correspond with the picture?” and three algebraic expressions, $a = b + c$, $b + c = a$, and $b = c + a$. These were presented as suggestions made by fictitious students. The purpose with the line segment was to symbolise unknown quantities, that is,

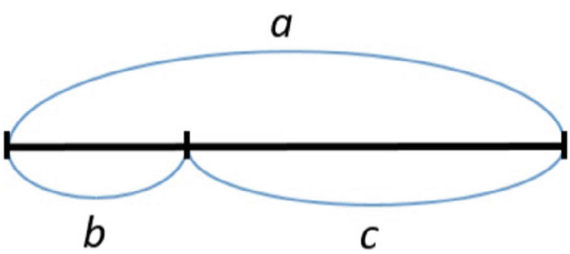

Fig. 15.7 Learning model used in grade 5 (inspired by Davydov et al., 2012). *Note.* Extracted from Eriksson et al. (2019)

Karim, Robin and Petra walk to school. Petra's distance to school is the same as Kim's and Robin's together.

$a = b + c$ $f + e = m$ $t = k - s$

Fig. 15.8 Problem situation presented in grade 5. *Note*. Extracted and translated from Eriksson et al. (2019)

to function as a learning model where structures and relations in the expressions were visualised and possible to elaborate on. The planned contradiction was that all suggestions from the fictitious students could be perceived as correct, since all three expressions had the same structure. The expressions consisted of the same variables, the same operator, and an equal sign, where the first two algebraic expressions matched the learning model while the third did not. The idea was that students would collectively use the line segments as a tool to visualise the inherent contradiction and discuss the meaning of the expressions. In the second sequence, in which students were to collectively work with the learning model, the teacher presented a problem situation in the form of a statement and three algebraic expressions (Fig. 15.8).

To offer different possibilities for collective discussions on algebraic structures, the variables in the expressions were symbolised with different letters. These expressions were also presented as suggestions from fictitious students. In the problem situation, a possible contradiction was planned for in the form of an additive character of the text as the word "together" usually signals that something should be added, but one of the algebraic expressions, $t = k - s$, was expressed as a subtraction.

A third sequence was distinguished when the teacher introduced a new expression that did not fit the given learning model but could fit the problem.

Previously in the lesson, a student, Matilda, asked if they could use the learning model presented earlier in the lesson (Fig. 15.7) when analysing and arguing why the fictional suggestion $a = b + c$ could correspond to the written information. She drew a learning model on the board (Fig. 15.9).

Another student, Nils, suggested that the expression $t = k - s$ also could correspond with the written information. He pointed to the longest line segment (a in Matilda's model) and said that it was s. He then pointed to the variable b and said that it could be k and to the variable c and said that it could be t. Nils finished his suggestion by saying, "so s minus k," The incorrect connection between the variables and the line segments and the fact that Nils said "s minus k" made Samuel object:

But now it's k minus s.

Excerpt 5

Samuel: But now it's k minus s [refers to the task].

Nils: Yes, but … [shrugs his shoulders].

Samuel: It’s the other way round.

Nils: Yes, exactly, the other way round.

Samuel: If you [inaudible] switch them it’s exactly so /…/ Well, if you take the shorter line segment, minus the longer line segment it will be negative.

Nils repeated what he had already said, and Samuel interrupted him and said that “if s is the longer line segment, then it is the shorter minus the longer line segment.” At this stage, the teacher chose to draw a similar model next to Matilda’s on the board and then asked Nils to write out the variables (see Fig. 15.10).

Samuel argued that the expression $t = k - s$ can be correct given that the longest line segment in the model is attributed to the variable k and the two shorter line segments s and t, but in the way Nils used the variables in the learning model they did not correspond with the expression. In response to Samuel’s objection, Nils shrugged his shoulders and said, “[y]es, but,” as if it did not matter that he reversed the variables in the expression. Through the objections, “[i]t’s the other way round” and “[i]f you switch them it’s exactly so,” Samuel expressed that the learning model was correct if k corresponded to the longer line segment and s to one of the shorter line segments. When Samuel said, “[w]ell, if you take the shorter line segment, minus the longer line segment, it will be minus,” he thus compared the lengths of the line segments and pointed out that a longer line segment cannot be subtracted from a shorter one without the answer being negative, which was visualised in the learning model.

Nils finally adjusted the learning model to match the expression and context (Fig. 15.11), and Samuel and the other students agreed.

By collectively testing different arguments in the learning model and adjusting the variables, the interrelationships of the variables and the meaning of subtraction were materialised for the class.

15.4.2.1 “Because it is the Same Letter”

Later in the lesson, the teacher presented another algebraic expression, $a = d + d$, in relation to the problem situation with the school route where the previously drawn learning model was on the board:

Excerpt 7

Teacher: If I were to write a different expression, then. If I were to write [writes $a = d + d$]. What would it mean in that case?

Filippo: Karim’s and Robin’s distances are the same.

Teacher: How do you know that?

Filippo: Because it’s the same letter.

Teacher:	Because it is the same letter and that would mean that it is the same? [Heard from the class: "I think so too"] What does Ezra say?
Ezra:	I say like Filippo, then it's exactly the same length of the line segments because it's … what's the word, the same letter.
	/…/
Teacher:	What does Sara say?
Sara:	Well … that, like, Karim's route is half, or Petra's route is twice as long as Karim's.

When the teacher wrote the new expression, she invited an extended reasoning on the relationship between whole and parts as double/half. The reasoning that took place thus enabled what appears to be a qualified algebraic understanding that $d + d = 2d$.

15.5 Discussion

The results are discussed below in relation to the objective, which was to exemplify the function of a learning model as a mediating tool for materialising algebraic thinking when students jointly discuss algebraic expressions.

At a general level, a learning model can enable students' theoretical work in whole-class discussions in its different phases: identifying problems, developing and substantiating their arguments, and making these arguments visible and making it possible for others to elaborate on them. With reference to Vygotsky's (1986) principle that thinking first occurs on a societal plane and later on an individual plane, the construction of a Zone of Proximal Development (ZPD) also needs to be understood as a collective activity (Arievitch, 2017; Carpay, n.d.). This can be seen as an opportunity for students to materialise and verbalise their algebraic thinking by integrating the models into their arguments (Broman et al., 2022; Davydov, 2008; Engeness, 2021; Gal'perin, 1969; Gorbov & Chudinova, 2000). When the learning models are available to the whole class, conditions are created for a collective reflection process (Zuckerman, 2004) where the students can test their ideas by exploring and transforming the model. Or to paraphrase Kieran and Dreyfus (1998), it is possible for the students to enter the thought universe of others. Such "collective algebraic thinking" creates possibilities for the students to work in a collective ZPD (Vygotsky, 1986).

The analysis indicates that the learning models enhanced the students' joint exploration of the idea behind algebraic expressions and thus materialised their algebraic thinking in the whole-class discussions in three ways: (1) materialising an argument, (2) materialising a problem, and (3) materialising a collective memory.

Materialising an Argument: With reference to the idea that learning activity should develop the students' reflective and critical thinking (Zuckerman, 2003, 2004), the results show that learning models can have such a function, that is, the tool contributes

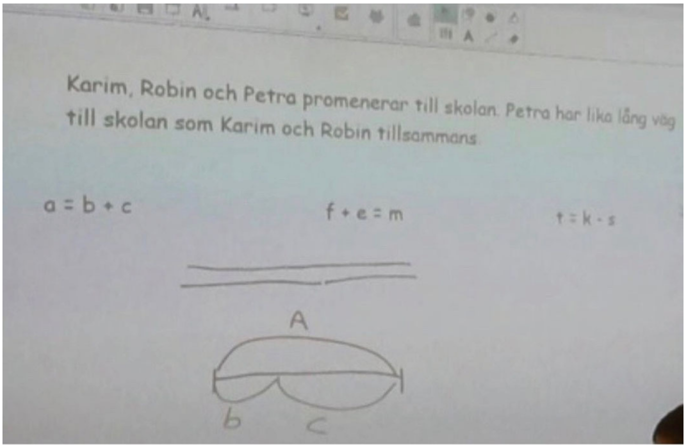

Fig. 15.9 Matilda's use of the learning model when providing an argument for the expression $a = b + c$

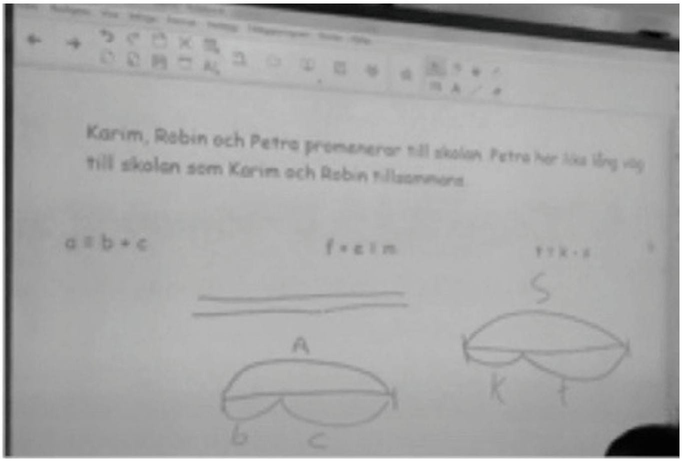

Fig. 15.10 Nils' understanding of the expression $t = k - s$ (on the right)

to promoting collective theoretical reflections (Davydov, 2008). The learning models in our study not only contributed to the qualification of the classroom discussions, they also were part of the common mathematical language of the class. In other words, they became part of the students' arguments by giving the arguments specific meanings that were available to other students in the class. The arguments were thus built up by verbal statements, gestures, and transformations of the learning model. In Excerpt 1, for example, Adam's moving of the rods together with the statement, "so you should do this," constituted a communicative whole that the other students could follow (cf., Roth & Radford, 2011).

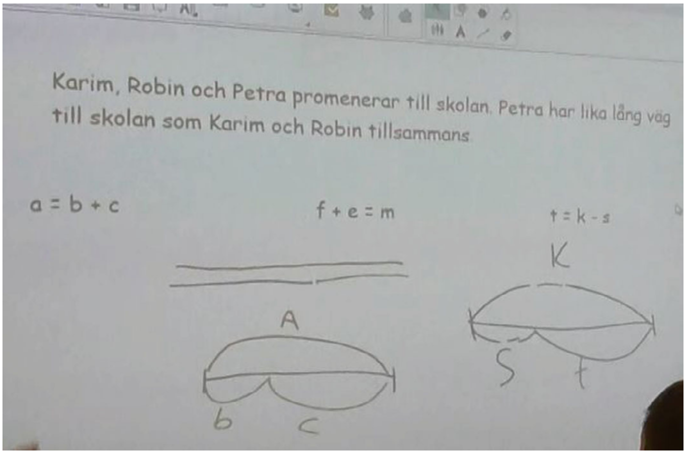

Fig. 15.11 Nils's adjustment of the learning model (on the right)

In grade 5 (Excerpt 5) when Samuel compared the lengths of the line segments in the learning model and pointed out that a longer line segment could not be subtracted from a shorter one without the answer being negative, he materialised the argument as to why $k - s$ is not the same as $s - k$.

Materialising a Problem: An obvious condition, with reference to Davydov's (2008) idea of ascending from the abstract to the concrete, is that the chosen learning models need to capture some of the central structures and relationships that the students are expected to discern, that is, the learning model needs to enable theoretical exploration. As such, the learning model enabled the testing and re-testing of arguments by visualising what could be a problem and what could be possible solutions. Like the situation in grade 1 where Adam and Elliot, together with the teacher (see Excerpt 1), discussed how rod construction number 6 (Fig. 15.4) was incorrect based on its incongruence with the original learning model on the board (Fig. 15.2). In so doing, they found that there was another, and mathematically more relevant, flaw with the construction—it did not show an equality. Having proved this they could use the learning model to propose solutions that would establish equality. In grade 5 (Excerpt 5), Nils' use of the variables s and k in the learning model did not correspond with the given expression $t = k - s$. The fact that Nils did not seem to discern the problem with how he had related the expression to the line segments in the learning model created a contradiction that triggered Samuel to qualify his arguments so Nils would understand what was problematic.

Materialising a Collective "Memory": When the teachers and students used the learning models to actively invite each other to share ideas not only verbally but also by writing them on a joint board, and not erasing anything without discussion, the learning models contributed to collectively materialising the modelling. This is what we have chosen to call a "collective memory." The classroom is often a complex situation with many participants, and where separate discussions and modelling can emerge simultaneously. Therefore, the simple use of a joint board in this structured way allows ideas and contributions to be active and visible for as long as necessary for the modelling to take place and become substantive in the collective work. The teacher can also direct the students’ focus to previously overlooked ideas, or to seemingly incongruent contributions. The idea of a collective memory thereby offers the teacher a tool for reviewing the discussion and steering it in constructive directions.

One example of this is when the students in grade 1 used the learning model that had been used earlier in the lesson and that was still available on the board (Fig. 15.2) to further the discussion towards manipulating the rod representation to fit the expression $c + c = a$. This version of the original expression, $c + b = a$, had been scrutinised in the mid-section of the lesson where the rod constructions (see Fig. 15.3) constituted the collective memory. Then, in Excerpt 3, we saw how Ines connected the adjusted expression to an adjusted version of the rod construction. Similarly, the discussion in grade 5 (see Excerpt 7) can be seen as an example of activating a collective memory. The learning models, which were available on a joint board (see Fig. 15.11), enabled the students to examine the new expression in relation to the previous reasoning even if the students did not draw an additional model. The visualisation of the different learning models and the different expressions allowed students to “remember” previous problems and solutions.

15.5.1 Concluding Remarks

In the research lessons we have analysed here, we have used a learning theory framework based on cultural historical and activity theory—learning activity (Davydov, 2008). In the planning of the lessons, in addition to the choice of learning models, a lot of work was put into designing the problem situations, contradictions, and thinking about how the chosen learning models would be introduced and used. Emphasis was also placed on creating the problem situations and trying to build in the contradictions or hooks that could create a motivation for the students to engage in the theoretical work that was sought. The inherent and emerging contradictions in the form of hooks or provocations also contributed to the need for students to assess the reasonableness of different proposed solutions and to anchor them mathematically (Toulmin, 2003). The teacher’s readiness to utilise emerging situations and then try to provoke the students to collectively qualify their reasoning was thus theoretically motivated (cf., Eriksson, 2017).

As noted by Ryve et al. (2013), among others, using mediating tools—here in the form of what Davydov (2008) refers to as learning models—provides better conditions for productive discussions. The examples of how to use learning models, we have given in this chapter, can hopefully contribute to teachers’ ambition to develop students’ algebraic thinking.

In this chapter, our main focus has been to exemplify how learning models can create conditions for students to follow, examine, and develop each other's arguments. However, it is important to note that the role of modelling within learning activity extends beyond these aspects. According to Gorbov and Chudinova (2000), modelling is a central component of learning activity since without modelling theoretical thinking, in our case algebraic thinking, is not possible (cf., Broman et al., 2022; Repkin, 2003b). Gorbov and Chudinova emphasise that the use of learning models needs to be qualified throughout students' educational journeys. Initially, modelling work focuses on understanding how concrete actions on an object can be modelled through learning models, such as, for example, line segments (see Fig. 15.1). This early stage also involves processing and transforming models. The structure of the object and the general mode of action are depicted in the model, incorporating fixed relations derived from the analysis of object actions rather than direct empirical experiences. As the students progress in the process of learning, the modelling becomes an integral part of their theoretical thinking, enabling them to engage in "reverse action" in reality, and acquire new knowledge about the relationships between original and replacement objects.

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